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One Step Hybrid Non-Linear Method for Stiff First Order Initial value Problems

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Abstract

This paper considers interpolation and collocation of rational approximate solution to give a continuous one step non-linear method for the solution of stiff initial value problems. The continuous method is evaluated at selected grid points to give discrete methods which are implemented in predictor-corrector method. The developed methods are found to be convergent and L-stable. Numerical results show that the method is efficient in handling stiff problems.

Keywords: Interpolation; collocation; rational approximate solution; non-linear method; continuous one step method; AMS subject classification: 65L05, 65L0D, 65D30

Introduction

Stiff systems of ODEs are very special cases of ODE, all the methods for approximating the solution to IVPs have error terms that involve higher derivative of the solution of the equation (Lambert and Mitchell, 1962). If the derivative can be reasonably bounded, then the method will have a predictable error bound that can be used to estimate the accuracy of the approximation. Even if the derivative grows as the steps increase, the error can be kept in relative control, provided that the solution also grows in magnitude. Problems mostly arise, however, when the magnitude of the derivative increases, but the solution does not. In this situation, the error can grow so large that it dominates the calculations. The IVPs for which this may likely occur are called stiff equations and are quite common particularly in the study of vibrations, chemical reactions and electrical circuits. Stiff systems derive their name from the motion of spring and mass systems that have large spring constant.

Consider the test equation

$$y'(t) = \lambda_i(t) \qquad y(0) = 1 \qquad (1)$$

for $\mathrm{Re}(\lambda_i) < 0$ decay exponentially fast as t increases, how a method performs on the test equation indicates how they will perform on more general equations (Shampine and Thompson, 2007). For instance, $\mathrm{Re}(\lambda_i) < 0$ for all eigenvalues, a commonly used stiffness index is given as

$$L = \max \left\| \mathrm{Re}(\lambda_i) \right\|$$

This measure is extended to general differential equations by considering eigenvalues of the local Jacobian matrix. Where L is not invariant under a simple rescaling of the problem. This raises the distinction between the mathematical problem and computational problem. The computational problem includes the nature of the error control and the error tolerances. In particular, rescaling a problem must include a corresponding change of error control if an equivalent problem is to be solved. An alternative measure is the ratio of the local solution time scale to the smallest damping time constant,

$$\min \left[\frac{-1}{\mathrm{Re}(\lambda_i)} \right]$$

This can be more useful when some $\mathrm{Re}(\lambda_i) < 0.$ The measure of stiffness is given by the stiffness ratio $S = \frac{\max\left\|\mathrm{Re}(\lambda_i)\right\|}{\min\left\|\mathrm{Re}(\lambda_i)\right\|}$ (Lambert and Mitchell, 1962).

In this, we construct a hybrid continuous method using Pade approximate solution which is implemented in predictor corrector method. The method is better than using polynomial basis function for reasons discussed earlier. Hybrid points and method of implementation is an advantage in the sense of better stability properties and increased order of the method with low truncation error. The computer code makes the results more interactive and not problem dependent.

$$y'(x) = f(x,y),\ y(x_n) = \eta_0,\ x_n \le x \le x_N \qquad (2)$$

where $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is a given real valued piecewise continuous function in the interval $x \in [x_n, x_N]$ and assumed to satisfies Lipchitz existence and uniqueness theorem.

Despite the success recorded by linear multistep method (LMM) in the numerical solution of initial value problems, most of the approaches failed when the problems are stiff, stiff oscillatory or singular problems. Adoption of rational approximate solution as basic functions which results into non-linear method have been effective in handling this setback, but their developments are tedious. The better efficiency lies in the fact that rational function of degree $n \times n$ usually produces or gives a better approximation than the Taylor polynomial of degree $2n$ (n is the order of the polynomial) (Gadella and Lara, 2013). Continuous formulation of method which enables evaluation at all points within the interval of integration has been well developed in construction of linear multistep method, but the application to construction of non-linear method has not been well established in literature, therefore, this research extends the continuous formulation of method to non-linear methods.

Lambert and Shaw (1965) represented an alternative procedure that was based on a local representation of the theoretical solution to first order initial value problem by specialized form of rational function

$$y(x) = \frac{p_n(x)}{b + x} \qquad (3)$$

where $p_n(x)$ is a polynomial of degree n. It was reported that this method can handle special singular initial value problems.

Luke *et al.* (1969) suggested approximate solution in the form

$$y(x) = \frac{\sum_{r=0}^{N} a_r x^r}{1 + \sum_{r=1}^{V} b_r x^r} \qquad (4)$$

to check the setback of (1), (2) is called the rational approximations or the Pade approximation. The resultant algorithms are non-linear methods which could cope with problems possessing singularities (Fatunla, 1988). Okosun and Ademuliyi (2007) presented a three step method for the numerical solution of ODEs with singularities, the scheme was based on rational functions approximation technique and their development analysis is based on power series expansion and Dalhquist stability test method which is valid to handle ODEs with singularities due to approximate solution used.

Mathematical Background

In this paper, we discuss the step by step approach in the derivation of continuous formulation method for the solution of stiff and singular problems. The idea is to approximate the solution $y(x)$ of (2) in the partition $\pi_{[a,b]} = [a = x_n < x_1 < \cdots < x_N = b]$ of the integration interval $[a,b]$ by Pade approximate solution in the form

$$y(x) = \frac{\sum_{i=0}^{m} a_i x^i}{1 + \sum_{p=1}^{k} b_p x^p} \qquad (5)$$

where $b_p, a_i \in \mathbb{R}$ are constants to be determined to the general first order initial value problems in the form (2).

Derivation of Methods

Let the approximate solution be given as Pade approximate solution in the form (5), $k + m = s + r - 1$, where r and s are numbers of interpolation and collocation points respectively. Interpolating (5), at x_{n+j}, $j = 0,1,2,\cdots,r$ and collocating the first derivative of (5), at x_{n+j}, $j = 1,2,\cdots,s$ gives a non-linear system of equations in the form

$$XA = U \qquad (6)$$

where

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_m & b_1 & \cdots & b_k \end{bmatrix}^T$$

$$U = \begin{bmatrix} y_n & y_{n+1} & \cdots & y_{n+r} & y'_{n+1} & \cdots & y'_{n+s} \end{bmatrix}^T$$

$$X = \begin{bmatrix} 1 & \cdots & x_n^m & -x_n y_n & \cdots & -x_n^k y_n \\ 1 & \cdots & x_{n+1}^m & -x_{n+1} y_{n+1} & \cdots & -x_{n+1}^k y_{n+1} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \cdots & x_{n+r}^m & -x_{n+r} y_{n+r} & \cdots & -x_{n+r}^k y_{n+r} \\ 0 & \cdots & m x_{n+1}^{m-1} & -\left(y_{n+1} + x_{n+1} y'_{n+1}\right) & \cdots & -\left(k y_{n+1} x_{n+1}^{k-1} + y'_{n+1} x_{n+1}^k\right) \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & m x_{n+s}^{m-1} & -\left(y_{n+s} + y'_{n+s} x_{n+s}\right) & \cdots & -\left(k y_{n+s} x_{n+s}^{k-1} + y'_{n+s} x_{n+s}^k\right) \end{bmatrix} \quad (7)$$

(7) is carefully chosen such that it gives a consistent solution. We then impose the following conditions on (5), and its first derivatives

$$y\left(x_{n+j}\right) = y_{n+j},\ j = 0,1,2,\cdots,r$$
$$y'\left(x_{n+j}\right) = f_{n+j},\ j = 0,1,2,\cdots,s \quad (8)$$

where r and s are the numbers of interpolation and collocation respectively. Solving (6) for the unknown parameters, using Crammer's method, substituting the results into (5), and after some algebraic sorting gives the continuous non-linear method which is evaluated at selected grid points to give discrete method which is implemented in predictor corrector method.

Analysis of the Method

Definition 1. Order of the Method:

Numerical analysis is not only the formulation of numerical methods, but also their analysis. Three central concepts in this analysis are convergence, rate of convergence and stability.

We associate the operator ℓ with the non-linear method defined by

$$\ell\left[y(x)\ :\ h\right] = y_{n+t} - y\left(x_{n+t}\right) = 0$$

where $y(x)$ is an arbitrary function continuously differentiable on $[a,b]$. Following Fatunla (1982), we can write terms in (7) as a Taylor series expansion about the point x to obtain the expansion

$$\ell\left[y(x)\ :\ h\right] = c_0 y(x) + c_1 h y'(x) + \ldots + c_p h^p y^p(x) + \ldots$$

where the constant coefficients, $c_p,\ p = 0,1,2,\ldots$ are given as

$$c_p = \frac{1}{p!}\left[\sum_{j=1}^{r} j^p \Phi_j - \frac{1}{(p-1)!}\sum_{j=1}^{r} j^{p-1}\Psi_j\right]$$

(7) has rate of convergence p if

$$\ell\left[y(x)\ :\ h\right] = o\left(h^{p+1}\right), c_0 = c_1 = \ldots = c_p = 0, c_{p+1} \neq 0$$

Therefore c_{p+1} is the error constant and $c_{p+1}h^{p+1}y^{p+1}$ is the local truncation error (LTE).

*Definition 2. Zero Stable (*Adesanya, Pantuvo and Umar, 2018*)*

Numerical method is said to be zero stable if

$$\lim_{h\to 0} y_{n+w} = y_n$$

where w is the evaluation point.

Definition 3. Consistent (Adesanya, Fotta and Abdulkadiri, 2015)

A numerical method is said to be consistent if

(i) it has rate of convergence $p \geq 1$

(ii) $\lim_{h\to 0}\left(\frac{1}{h}\left(y_{n+w} - y_n\right)\right) = w y'_n$

Definition 4. Convergent (Adesanya, Fotta and Abdulkadiri, 2015)

A numerical method is said to be convergent

(i) $\lim_{h\to 0}\left(y(x)-y_n(x)\right)\to 0$ where $y(x)$ is the exact solution and $y_n(x)$ is the approximate solution.

(ii) it is consistent and zero stable.

Definition 5. Region of Absolute Stability (Sunday, Adesanya and Odekunle, 2014)

Region of absolute stability is a region in the complex $z=\lambda h$ plane, where $z=\lambda h$. It is defined as those values of z such that the numerical solutions of $y'=-\lambda y$ satisfy $j\to\infty$ as $j\to\infty$

Definition 6 A-Stable (Adesanya, Pantuvo and Umar, 2018)

A numerical method is said to be A-stable if

$$\lim_{z\to\infty} R(z)\le 1$$

Definition 7. L-Stability (Philippe and Bernard, 2006)

A numerical method is said to be L-stable if

(i) it is A-stable

(ii) $\lim_{z\to\infty}\left|R(z)\right|\to 0$

where $R(z)$ is called the stability polynomial of the method.

Specification of the Method

The points for interpolation and collocation is shown in Table 1.

Table 1: Interpolation and Collocation Points

Method	IP	CP	EP	AS
Corrector y_{n+1}	$0,u$	u,v	1	$\frac{a_0+a_1x}{1+b_1x+b_2x^2}$
Predictor for y_{n+v}	$0,u$	u	v	$\frac{a_0+a_1x}{1+b_1x}$
Predictor for y_{n+u}	0	0	u	$\frac{a_0}{1+b_1x}$

IP means Interpolation points; CP means Collocation points; EP means Evaluation points; AS means Approximate solution

Results of the Corrector Constants

$$A_0=\left(\begin{array}{c}
x_n^2y_{n+u}^2+h^2u^2y_{n+u}^2-x_n^2y_{n+u}y_{n+v}-x_n^2y_ny_{n+u}+x_n^2y_ny_{n+v}\\
+2hux_ny_{n+u}^2-h^3u^3f_{n+u}y_{n+v}-h^2u^2y_{n+u}y_{n+v}+h^4u^2v^2f_{n+u}f_{n+v}\\
-h^2u^2x_ny_nf_{n+u}+h^2v^2x_ny_nf_{n+v}-h^2u^2x_n^2f_{n+u}f_{n+v}-2hux_ny_ny_{n+u}\\
+2hvx_ny_ny_{n+v}+hux_n^2f_{n+u}y_{n+v}-hvx_n^2f_{n+v}y_{n+u}-h^4u^3vf_{n+u}f_{n+v}\\
+2h^3u^2vf_{n+u}y_{n+v}-h^3u^2vf_{n+v}y_{n+u}-hux_n^2y_nf_{n+u}+hvx_n^2y_nf_{n+v}\\
-h^3u^3x_nf_{n+u}f_{n+v}-h^2u^2x_nf_{n+v}y_{n+u}-h^2v^2x_nf_{n+v}y_{n+u}-2hvx_ny_{n+u}y_{n+v}\\
+2h^2uvx_nf_{n+u}y_{n+v}+h^2uvx_n^2f_{n+u}f_{n+v}+h^3uv^2x_nf_{n+u}f_{n+v}
\end{array}\right)$$

$$A_1 = h\begin{pmatrix} hu^2 y_n y_{n+u}^2 + 2ux_n y_{n+u}^2 y_{n+v} - 2vx_n y_{n+u}^2 y_{n+v} - 2ux_n y_n y_{n+u} y_{n+v} \\ +2vx_n y_n y_{n+u} y_{n+v} - hu^2 y_n y_{n+u} y_{n+v} - hu^2 x_n f_{n+v} y_{n+u}^2 \\ -hv^2 x_n f_{n+v} y_{n+u}^2 - h^2 u^3 y_n f_{n+u} y_{n+v} + h^3 u^2 v^2 y_n f_{n+u} f_{n+v} \\ +2huvx_n f_{n+v} y_{n+u}^2 - hu^2 x_n y_n f_{n+u} y_{n+v} + hv^2 x_n y_n f_{n+v} y_{n+u} \\ -h^3 u^3 v y_n f_{n+u} f_{n+v} + 2h^2 u^2 v y_n f_{n+u} y_{n+v} \\ -h^2 u^2 v y_n f_{n+v} y_{n+u} + h^2 u v^2 x_n y_n f_{n+u} f_{n+v} \\ -h^2 u^2 v x_n y_n f_{n+u} f_{n+v} + 2huvx_n y_n f_{n+u} y_{n+v} - 2huvx_n y_n f_{n+v} y_{n+u} \end{pmatrix}$$

$$A_2 = h\begin{pmatrix} -2uy_{n+u}^2 y_{n+v} + 2vy_{n+u}^2 y_{n+v} + hu^2 f_{n+v} y_{n+u}^2 + hv^2 f_{n+v} y_{n+u}^2 \\ +2uy_n y_{n+u} y_{n+v} - 2vy_n y_{n+u} y_{n+v} - 2huvf_{n+v} y_{n+u}^2 + hu^2 y_n f_{n+u} y_{n+v} \\ -hv^2 y_n f_{n+v} y_{n+u} - h^2 uv^2 y_n f_{n+u} f_{n+v} + h^2 u^2 v y_n f_{n+u} f_{n+v} \\ -2huvy_n f_{n+u} y_{n+v} + 2huvy_n f_{n+v} y_{n+u} \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -2x_n y_{n+u}^2 - 2huy_{n+u}^2 + 2x_n y_n y_{n+u} - 2x_n y_n y_{n+v} + 2x_n y_{n+u} y_{n+v} \\ +h^2 u^2 y_n f_{n+u} - h^2 v^2 y_n f_{n+v} + 2huy_n y_{n+u} - 2hvy_n y_{n+v} + h^3 u^3 f_{n+u} f_{n+v} \\ +h^2 u^2 f_{n+v} y_{n+u} + h^2 v^2 f_{n+v} y_{n+u} + 2hvy_{n+u} y_{n+v} - 2h^2 uvf_{n+u} y_{n+v} \\ +2hux_n y_n f_{n+u} - 2hvx_n y_n f_{n+v} - h^3 uv^2 f_{n+u} f_{n+v} + 2h^2 u^2 x_n f_{n+u} f_{n+v} \\ -2hux_n f_{n+u} y_{n+v} + 2hvx_n f_{n+v} y_{n+u} - 2h^2 uvx_n f_{n+u} f_{n+v} \end{pmatrix}$$

$$A_4 = \begin{pmatrix} -y_n y_{n+u} + y_n y_{n+v} - y_{n+u} y_{n+v} + y_{n+u}^2 - huy_n f_{n+u} + hvy_n f_{n+v} \\ -h^2 u^2 f_{n+u} f_{n+v} + huf_{n+u} y_{n+v} - hvf_{n+v} y_{n+u} + h^2 uvf_{n+u} f_{n+v} \end{pmatrix}$$

Where

$$a_0 = \frac{A_1}{A_0}, a_1 = \frac{A_2}{A_0}, b_1 = \frac{A_3}{A_0}, b_2 = \frac{A_4}{A_0}$$

Results for y_{n+v} constants

$$B_0 = \left(x_n y_{n+u} - x_n y_n - h^2 u^2 f_{n+u} - hux_n f_{n+u}\right)$$

$$B_1 = \left(x_n y_{n+u}^2 - x_n y_n y_{n+u} - h^2 u^2 y_n f_{n+u} - hux_n y_n f_{n+u}\right)$$

$$B_2 = \left(-y_{n+u}^2 + y_n y_{n+u} + huy_n f_{n+u}\right)$$

$$B_3 = \left(y_n - y_{n+u} + huf_{n+u}\right)$$

where

$$a_0 = \frac{B_1}{B_0}, a_1 = \frac{B_2}{B_0}, b_1 = \frac{B_3}{B_0}$$

Results for y_{n+u} constants

$$C_0 = \left(-x_n f_{n+u} - y_{n+u} - huf_{n+u}\right)$$

$$C_1 = \left(-y_n y_{n+u} - huy_n f_{n+u}\right)$$

$$C_2 = f_{n+u}$$

where

$$a_0 = \frac{C_1}{C_0}, b_1 = \frac{C_2}{C_0}$$

Results of the Continuous Method

$$y^c_{n+t} = \frac{\left[\begin{array}{c} u^2 y_n y^2_{n+u} - 2tuy^2_{n+u}y_{n+v} + 2tvy^2_{n+u}y_{n+v} - u^2 y_n y_{n+u} y_{n+v} \\ -hu^3 y_n f_{n+u} y_{n+v} + htv^2 f_{n+v} y^2_{n+u} + htu^2 y^2_{n+u} y'_{n+v} + 2tuy_n y_{n+u} y_{n+v} \\ -2tvy_n y_{n+u} y_{n+v} - 2htuvf_{n+v} y^2_{n+u} + htu^2 y_{n+u} y_{n+v} y'_{n+u} + h^2u^2v^2 y_n f_{n+u} f_{n+v} \\ +htu^2 y_n f_{n+u} y_{n+v} - htv^2 y_n f_{n+v} y_{n+u} + 2hu^2 vy_n f_{n+u} y_{n+v} - hu^2 vy_n f_{n+v} y_{n+u} \\ -h^2u^3 vy_n f_{n+u} f_{n+v} - htu^2 f_{n+u} y_{n+u} y_{n+v} + h^2tuv^2 f_{n+u} f_{n+v} y_{n+u} \\ -h^2tu^2 vf_{n+u} f_{n+v} y_{n+u} + 2htuvf_{n+u} y_{n+u} y_{n+v} - h^2tuv^2 f_{n+v} y_{n+u} y'_{n+u} \\ +h^2tu^2 vf_{n+v} y_{n+u} y'_{n+u} - 2htuvy_{n+u} y_{n+v} y'_{n+u} - h^2tuv^2 y_n f_{n+u} f_{n+v} \\ +h^2tu^2 vy_n f_{n+u} f_{n+v} - 2htuvy_n f_{n+u} y_{n+v} + 2htuvy_n f_{n+v} y_{n+u} \end{array}\right]}{\left[\begin{array}{c} t^2 y^2_{n+u} + u^2 y^2_{n+u} - t^2 y_{n+u} y_{n+v} - u^2 y_{n+u} y_{n+v} - 2tuy^2_{n+u} - t^2 y_n y_{n+u} \\ +t^2 y_n y_{n+v} + 2tuy_n y_{n+u} - 2tvy_n y_{n+v} - hu^3 f_{n+u} y_{n+v} + 2tvy_{n+u} y_{n+v} \\ +h^2u^2v^2 f_{n+u} f_{n+v} - htu^2 f_{n+u} y_{n+u} + ht^2 uf_{n+u} y_{n+u} + htv^2 f_{n+v} y_{n+u} \\ -ht^2 vf_{n+v} y_{n+u} + 2hu^2 vf_{n+u} y_{n+v} - hu^2 vf_{n+v} y_{n+u} - h^2t^2u^2 f_{n+u} y'_{n+v} \\ +htu^2 y_{n+u} y'_{n+u} - ht^2 uy_{n+u} y'_{n+u} + htu^2 y_{n+u} y'_{n+v} + ht^2 uy_{n+v} y'_{n+u} \\ +htu^2 y_n f_{n+u} - ht^2 uy_n f_{n+u} - htv^2 y_n f_{n+v} + ht^2 vy_n f_{n+v} - 2htuvy_{n+v} y'_{n+u} \\ -h^2u^3 vf_{n+u} f_{n+v} + h^2tu^3 f_{n+u} y'_{n+v} - h^2tuv^2 f_{n+v} y'_{n+u} + h^2t^2uvf_{n+v} y'_{n+u} \end{array}\right]}$$

$$y^p_{n+t} = \left(h \frac{ty^2_{n+u} - ty_n y_{n+u} + hu^2 y_n f_{n+u} - htuy_n f_{n+u}}{-y_n + y_{n+u} + h^2u^2 f_{n+u} - huf_{n+u}} \right)$$

$$y^p_{n+t} = \left(y_n \frac{y_{n+u}}{y_{n+u} - htf_n} \right)$$

where $t = \frac{x - x_n}{h}$. Evaluating the continuous method as stated in Table 1 gives

$$y_{n+1} = -\frac{\left[\begin{array}{c} u^2 y_n y_{n+u}^2 - 2u y_{n+u}^2 y_{n+v} + 2v y_{n+u}^2 y_{n+v} - u^2 y_n y_{n+u} y_{n+v} \\ +hu^2 y_{n+u}^2 f_{n+v} + hv^2 y_{n+u}^2 f_{n+v} + 2u y_n y_{n+u} y_{n+v} - 2v y_n y_{n+u} y_{n+v} \\ -2huv y_{n+u}^2 f_{n+v} + hu^2 y_n y_{n+v} f_{n+u} - hu^3 y_n y_{n+v} f_{n+u} \\ -hv^2 y_n y_{n+u} f_{n+v} + 2huv y_n y_{n+u} f_{n+v} - 2huv y_n y_{n+v} f_{n+u} \\ -h^2 uv^2 y_n f_{n+u} f_{n+v} + h^2 u^2 v y_n f_{n+u} f_{n+v} - h^2 u^3 v y_n f_{n+u} f_{n+v} \\ -hu^2 v y_n y_{n+u} f_{n+v} + 2hu^2 v y_n y_{n+v} f_{n+u} + h^2 u^2 v^2 y_n f_{n+u} f_{n+v} \end{array}\right]}{\left[\begin{array}{c} y_n y_{n+u} - y_n y_{n+v} + 2u y_{n+u}^2 + y_{n+u} y_{n+v} - u^2 y_{n+u}^2 - y_{n+u}^2 \\ -2u y_n y_{n+u} + 2v y_n y_{n+v} + u^2 y_{n+u} y_{n+v} - 2v y_{n+u} y_{n+v} \\ +h^2 u^3 v f_{n+u} f_{n+v} + hu y_n f_{n+u} - hv y_n f_{n+v} - hu^2 y_{n+u} f_{n+v} \\ +hu^3 y_{n+v} f_{n+u} - hv^2 y_{n+u} f_{n+v} - hu^2 y_n f_{n+u} + hv^2 y_n f_{n+v} \\ -hu y_{n+v} f_{n+u} + hv y_{n+u} f_{n+v} + h^2 u^2 f_{n+u} f_{n+v} - h^2 u^3 f_{n+u} f_{n+v} \\ +hu^2 v y_{n+u} f_{n+v} - 2hu^2 v y_{n+v} f_{n+u} - h^2 u^2 v^2 f_{n+u} f_{n+v} \\ -h^2 uv f_{n+u} f_{n+v} + 2huv y_{n+v} f_{n+u} + h^2 uv^2 f_{n+u} f_{n+v} \end{array}\right]} \tag{9}$$

$$y_{n+v} = \left(\frac{v y_{n+u}^2 - v y_n y_{n+u} + hu^2 y_n f_{n+u} - huv y_n f_{n+u}}{-v y_n + v y_{n+u} + hu^2 f_{n+u} - huv f_{n+u}} \right) \tag{10}$$

$$y_{n+u} = \left(\frac{y_n y_{n+u} + hu y_n f_{n+u}}{y_{n+u}} \right) \tag{11}$$

Results of the Discrete Methods

We consider following cases

Case 1: We considered an equal interval method, that is $u = \frac{1}{3}, v = \frac{2}{3}$

$$y_{n+1} = \left(\frac{\begin{array}{c} 54 y_{n+\frac{1}{3}}^2 y_{n+\frac{2}{3}} + 9 y_n y_{n+\frac{1}{3}}^2 + 9h f_{n+\frac{2}{3}} y_{n+\frac{1}{3}}^2 - 63 y_n y_{n+\frac{1}{3}} y_{n+\frac{2}{3}} \\ -4h^2 y_n f_{n+\frac{1}{3}} f_{n+\frac{2}{3}} - 18h y_n f_{n+\frac{1}{3}} y_{n+\frac{2}{3}} - 6h y_n f_{n+\frac{2}{3}} y_{n+\frac{1}{3}} \end{array}}{\begin{array}{c} -27 y_n y_{n+\frac{1}{3}} - 27 y_n y_{n+\frac{2}{3}} + 18 y_{n+\frac{1}{3}} y_{n+\frac{2}{3}} + 36 y_{n+\frac{1}{3}}^2 \\ -18h y_n f_{n+\frac{1}{3}} + 18h y_n f_{n+\frac{2}{3}} + 2h^2 f_{n+\frac{1}{3}} f_{n+\frac{2}{3}} - 15h f_{n+\frac{2}{3}} y_{n+\frac{1}{3}} \end{array}} \right)$$

$$y_{n+\frac{2}{3}} = \left(\frac{6 y_n y_{n+\frac{1}{3}} - 6 y_{n+\frac{1}{3}}^2 + h y_n f_{n+\frac{1}{3}}}{6 y_n - 6 y_{n+\frac{1}{3}} + h f_{n+\frac{1}{3}}} \right)$$

$$y_{n+\frac{1}{3}} = \left(\frac{y_n y_{n+\frac{1}{3}} + \frac{1}{3} h y_n f_{n+\frac{1}{3}}}{y_{n+\frac{1}{3}}} \right)$$

Case 2: We consider an interval where the hybrid points are at middle points, that is, $u = \frac{2}{5}, v = \frac{3}{5}$

$$y_{n+1}=\left(\frac{\begin{gathered}250y_{n+\frac{2}{5}}^{2}y_{n+\frac{3}{5}}+100y_{n}y_{n+\frac{2}{5}}^{2}+25hf_{n+\frac{3}{5}}y_{n+\frac{2}{5}}^{2}-350y_{n}y_{n+\frac{2}{5}}y_{n+\frac{3}{5}}\\-18h^{2}y_{n}f_{n+\frac{2}{5}}f_{n+\frac{3}{5}}-120hy_{n}f_{n+\frac{2}{5}}y_{n+\frac{3}{5}}+15hy_{n}f_{n+\frac{3}{5}}y_{n+\frac{2}{5}}\end{gathered}}{\begin{gathered}-125y_{n}y_{n+\frac{2}{5}}-125y_{n}y_{n+\frac{3}{5}}+25y_{n+\frac{2}{5}}y_{n+\frac{3}{5}}+225y_{n+\frac{2}{5}}^{2}-150hy_{n}f_{n+\frac{2}{5}}\\+150hy_{n}f_{n+\frac{3}{5}}+12h^{2}f_{n+\frac{2}{5}}f_{n+\frac{3}{5}}+30hf_{n+\frac{2}{5}}y_{n+\frac{3}{5}}-110hf_{n+\frac{3}{5}}y_{n+\frac{2}{5}}\end{gathered}}\right)$$

$$y_{n+\frac{3}{5}}=\left(\frac{15y_{n}y_{n+\frac{2}{5}}-15y_{n+\frac{2}{5}}^{2}+2hy_{n}f_{n+\frac{2}{5}}}{15y_{n}-15y_{n+\frac{2}{5}}+2hf_{n+\frac{2}{5}}}\right)$$

$$y_{n+\frac{2}{5}}=\left(\frac{y_{n}y_{n+\frac{2}{5}}+\frac{2}{5}hy_{n}f_{n+\frac{2}{5}}}{y_{n+\frac{2}{5}}}\right)$$

Case 3: We consider an interval where the hybrid points moves closer to the grid points, $u=\frac{3}{10}, v=\frac{7}{10}$

$$y_{n+1}=\left(\frac{\begin{gathered}4000y_{n+\frac{3}{10}}^{2}y_{n+\frac{7}{10}}+450y_{n}y_{n+\frac{3}{10}}^{2}+800hf_{n+\frac{7}{10}}y_{n+\frac{3}{10}}^{2}\\-4450y_{n}y_{n+\frac{3}{10}}y_{n+\frac{7}{10}}-294h^{2}y_{n}f_{n+\frac{3}{10}}f_{n+\frac{7}{10}}\\-1155hy_{n}f_{n+\frac{3}{10}}y_{n+\frac{7}{10}}-665hy_{n}f_{n+\frac{7}{10}}y_{n+\frac{3}{10}}\end{gathered}}{\begin{gathered}-2000y_{n}y_{n+\frac{3}{10}}-2000y_{n}y_{n+\frac{7}{10}}+1550y_{n+\frac{3}{10}}y_{n+\frac{7}{10}}\\+2450y_{n+\frac{3}{10}}^{2}-1050hy_{n}f_{n+\frac{3}{10}}+1050hy_{n}f_{n+\frac{7}{10}}\\+126h^{2}f_{n+\frac{3}{10}}f_{n+\frac{7}{10}}-105hf_{n+\frac{3}{10}}y_{n+\frac{7}{10}}-915hf_{n+\frac{7}{10}}y_{n+\frac{3}{10}}\end{gathered}}\right)$$

$$y_{n+\frac{7}{10}}=\left(\frac{35y_{n}y_{n+\frac{3}{10}}-35y_{n+\frac{3}{10}}^{2}+6hy_{n}f_{n+\frac{3}{10}}}{35y_{n}-35y_{n+\frac{3}{10}}+6hf_{n+\frac{3}{10}}}\right)$$

$$y_{n+\frac{3}{10}}=\left(\frac{y_{n}y_{n+\frac{3}{10}}+\frac{3}{10}hy_{n}f_{n+\frac{3}{10}}}{y_{n+\frac{3}{10}}}\right)$$

Results of the Stability Properties

Substituting the test equation $y'=\lambda y$ into $(9),(10)$ and (11), gives the stability function as shown in Table 2 .

Table 2: Stability properties

	Order	LTE	Cons	$R(z)$	ZS	S
y_{n+1}	3	$\frac{1}{36}\frac{\begin{pmatrix}\left(-9(y_n'')^3-2(y_n''')^2y_n+12y_n'y_n''y_n'''\right)\\(u-1)^2\left(-u+3v+2uv-4v^2\right)\end{pmatrix}}{(u-3v)\left(2(y_n')^2-y_n''y_n\right)}$	y_n'	$\frac{\begin{pmatrix}u-v+u^2z-v^2z\\-u^2z^2-uz+2vz+uvz^2-uv^2z^2+u^2vz^2\end{pmatrix}}{\begin{pmatrix}(z-uz-1)\\\left(-v-v^2z-uz+2vz+uvz\right)\end{pmatrix}}$	y_n	L
y_{n+v}	2	$\frac{1}{12}v(u-v)^2\frac{-3(y_n'')^2+2y_n'y_n'''}{y_n'}$	vy_n'	$\left(\frac{uz+1}{-vz^2+uz+1}\right)$	y_n	L
y_{n+u}	1	$\frac{1}{2}u^2\frac{2(y_n')^2-y_n''y_n}{y_n}$	uy_n'	$(zu+1)$	y_n	L

where

Cons $-$ results of consistency i.e. $\lim_{h\to 0}\frac{1}{h}\left(y_{n+j}-y_n\right)$, $R(z)-$ stability function, ZS $-$ results of zero stability i.e. $\lim_{h\to 0}\left(y_{n+j}\right)$, S $-$ stability, L $-$ L stable, LTE $-$ results of the local truncation error.

Numerical Examples

We consider the following problems to test the efficiency of the developed method.

Example 1: We consider the non-linear stiff system

$$\begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix} = \begin{pmatrix} -1002y_1(x)+1000y_2(x) \\ y_1(x)-y_2(x)\left(1+y_2(x)\right) \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$h = 0.01$ on $0 \le t \le \tau.$ The smaller the t, the more serious the stiffness of the system.

The exact solution

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} \exp(-2x) \\ \exp(-x) \end{pmatrix}$$

Source: Yatim, Ibrahim, Othman, and Suleiman (2013)

Example 2: We consider a singular problem

$$y'(x) = 1 + y^2(x), y(0) = 1$$

with the exact solution

$$y(x) = \tan\left(x + \frac{\pi}{4}\right)$$

with singularities at $x = -\frac{\pi}{4}$ within the interval $0 \le x \le 1$

Source: Fatunla (1982)

The following notations are used in the tables $abs\left(y - y_n\right)_i$ is the absolute error for case $i = 1, 2, 3$ and $tt(-\tau\tau) = tt \times 10^{-\tau\tau}$

Table 3: Results for Example I

N	y_i	*Exact*	*Computed*	$(\lvert y-y_n \rvert)_1$	*Computed*	$(\lvert y-y_n \rvert)_2$	*Computed*	$(\lvert y-y_n \rvert)_3$	*Yatim*(2013)
5	y_1	$4.5400e-05$	$4.5495e-05$	$[9.4719e-08]$	$4.5457e-05$	$[5.6741e-08]$	$4.5514e-05$	$[1.1374e-07]$	$2.5736e-04$
	y_2	$6.7380e-03$	$6.7450e-03$	$[7.0243e-06]$	$6.7422e-03$	$[4.208e-06]$	$6.7464e-03$	$[8.4341e-06]$	$5.2000e-03$
10	y_1	$2.0612e-09$	$2.0698e-09$	$[8.6012e-12]$	$2.0663e-09$	$[5.1551e-12]$	$2.0715e-09$	$[1.0330e-11]$	$1.1000e-03$
	y_2	$4.5400e-05$	$4.5495e-05$	$[9.4719e-08]$	$4.5457e-05$	$[5.6741e-08]$	$4.5514e-05$	$[1.1374e-07]$	$3.7659e-04$
15	y_1	$9.3576e-14$	$9.4164e-14$	$[5.8622e-16]$	$9.3928e-14$	$[3.5130e-16]$	$9.4280e-14$	$[7.0412e-16]$	$8.5506e-05$
	y_2	$3.0590e-07$	$3.0686e-07$	$[9.5664e-10]$	$3.0648e-07$	$[5.7362e-10]$	$3.0705e-07$	$[1.1487e-09]$	$6.9774e-05$
20	y_1	$4.2484e-18$	$4.2839e-18$	$[3.5521e-20]$	$4.2696e-18$	$[2.1276e-20]$	$4.2910e-18$	$[4.2671e-20]$	$3.2882e-08$
	y_2	$2.0612e-09$	$2.0698e-09$	$[8.5985e-12]$	$2.0663e-09$	$[5.1546e-12]$	$2.0715e-09$	$[1.0325e-11]$	$1.0790e-06$

Table 4: Results for Example VII at $h = 0.00125$

x	*Exact*	*Computed*	$(\lvert y-y_n \rvert)_1$	*Computed*	$(\lvert y-y_n \rvert)_2$	*Computed*	$(\lvert y-y_n \rvert)_3$	*Fatunla*(1982)
0.1	$1.2230e+00$	$1.2231e+00$	$[6.1568e-05]$	$1.2231e+00$	$[2.8738e-05]$	$1.2231e+00$	$[7.6775e-05]$	$[1.2281e-02]$
0.3	$1.8958e+00$	$1.8959e+00$	$[1.5689e-04]$	$1.8957e+00$	$[3.8772e-05]$	$1.8960e+00$	$[2.4915e-04]$	$[5.5799e-02]$
0.5	$3.4082e+00$	$3.4084e+00$	$[2.1506e-04]$	$3.4073e+00$	$[9.4300e-04]$	$3.4090e+00$	$[7.7232e-04]$	$[2.1342e-01]$
0.7	$1.1681e+01$	$1.1674e+01$	$[6.5021e-03]$	$1.1646e+01$	$[3.5201e-02]$	$1.1689e+01$	$[7.5653e-03]$	$[3.0917e+00]$
0.9	$-8.6876e+00$	$-8.6246e+00$	$[6.3074e-02]$		$-$		$-$	

Discussion of results

We observed that at $N=20$ in example 1, the error of y_1 and y_2 in case 1 are $2.7791e-20$ and $3.002e-22$ respectively, while those of case 2 and case 3 for y_1 and y_2 are $3.8982e-19$, $4.0934e-21$ and $1.7305e-19, 1.8147e-21$ respectively. Likewise, For singular problem, Table 4 shows clearly that case 1 method is suitable in handling problems with singularity since the other two cases fail to give results after the point of singularity as shown in Table 4.

Moreover, the methods developed are of other three, consistent, convergent, zero stable, linearly stable. They are explicit method and performs well for stiff problem which contradicts existing literature that explicit method cannot handle stiff problem effectively, hence the methods derived are efficient and computationally reliable.

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