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INVESTIGATING THE CONCEPT OF MIRROR SYMMETRY AND THE K-THEORY OF P-ADIC GROUPS

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Abstract
The relationship between mirror symmetry and the K-theory associated with p-adic groups, seems to be neglected and distant. This paper explores a deep connection between mirror symmetry and the k-theory related to p-adic groups. Originating from string theory, mirror symmetry provides significant revelations about the geometry of complex manifolds. In both algebraic topology and algebraic geometry, k-theory is presented as an influential tool, where the study investigates how mirror symmetry, primarily used in the context of complex varieties, can be modified for application to p-adic groups. By delving into the essential structures and relationships, this research presented a novel perspective on how arithmetic geometry is related to theoretical physics. The results of this investigation deepen the understanding of the algebraic and geometric properties of p-adic groups, with ramifications for number theory, p-adic approximations of vertex functions, representation theory, and the area of geometric engineering.

Keywords: Symmetry, Group theory, K-theory, P-adic group, Algebraic geometry, Theoretical physics.

Introduction

Mirror symmetry is an innovative idea that has changed the field of contemporary theoretical physics and mathematics. It arose from the challenges posed by the long-standing issue of Calabi-Yau manifolds in string theory, and its significant consequences have echoed across numerous fields such as algebraic geometry, topology, and quantum field theory. Essentially, mirror symmetry reveals a profound duality between what appears to be different mathematical and physical phenomena (Wenzhe, 2023).

Recognizing the significance of Yang-Mills gauge theories in explaining particle, interactions and their role in vector bundle, mathematics has stimulated a renewed collaboration between these two domains. This has led to an increased appreciation for concepts such as instantons and monopoles from both physical and mathematical perspectives. The advent of supersymmetry, alongside its logical expansion into superstring theory, has further broadened the interplay between physics and mathematics, fostering deep exchanges that persist to this day (Aharony, et al, 2023).

Fundamental inquiries within one discipline frequently mirror those found in the other. However, even presently, physicists and mathematicians often struggle to communicate and collaborate effectively. This appears to stem from two primary issues. Firstly, the jargon employed in both fields is significantly different. This communication barrier is gradually being bridged as both communities recognize the importance of being "bilingual". The second, more profound issue arises from the differing approaches to scientific methodology between the two realms.

Mathematicians prioritize a rigorous foundation and the interplay of various structures, while physicists focus more on the clarity of physical concepts and the interrelationships of ideas, sometimes sacrificing mathematical precision in the process. This divergence can create tension, as mathematicians respect physicists for their intuition yet remain skeptical of their results, leading them to establish rigorous frameworks to substitute for the underlying physical reasoning. Conversely, physicists, now acknowledging modern mathematics' critical role in theoretical physics, feel that striving for increased rigor—while admirable—can divert attention from their primary objective of understanding the natural world. Consequently, we are positioned at a precarious moment in the joint history of these fields which, despite their mutual reliance, sometimes exhibits dysfunctional interdependence (Treumann, 2019).

Mirror symmetry exemplifies a larger concept known as duality, showcasing how different physical systems can display a sophisticated isomorphic relationship. This intricate isomorphism mainly stems from the effect of quantum corrections. A useful analogy can be drawn with Fourier transforms, where localized elements, like products, relate to convolution products that require integration throughout the entire space. This relationship enhances our understanding of isomorphisms within conventional frameworks. In this scenario, complex quantities associated with quantum changes in one system are connected to more basic classical issues in another. Therefore, recognizing these dualities helps transform difficult physical challenges into more manageable versions within the dual structure. This emphasizes the important developments realized in our grasp of quantum theories and string theory, which are linked to the awareness of duality symmetries (Wenzhe, 2023).

Although we recognize that our grasp of why duality symmetries are prevalent in physics remains vague, and we do not possess conclusive evidence for their existence in some cases, numerous compelling validations have strengthened confidence in their importance. Still, these validations do not adequately elucidate the underlying mechanisms of duality symmetries. The only intuitive explanation we currently have for duality is the idea of 'scarcity of rich structures,' since credible quantum theories are often quite intricate. Hence, different methods aimed at developing comparable quantum systems generally lead to equivalent results (Andrey, 2022).

Nevertheless, mirror symmetry is a notable exception, as we have a relatively clear understanding of its functionality. Additionally, a mathematical framework has been developed to formalize many assertions emerging from physical theories, contributing to the field's significant advancement.

Our objective is to explore this complex duality that fascinates both physicists and mathematicians, aiming to motivate additional inquiries into this duality and possibly shed light on other dualities while strengthening the connection between physics and mathematics (Aspinwall, 2009).

Algebraic K-theory consists of two main components: the classical theory, which examines the Grothendieck group of a category along with its essential algebraic characteristics, and the higher algebraic K-theory, which advances through methods derived from topology or homology (Iritani, 2011). The Grothendieck group is classified into three primary categories. The first category involves group completion, which is vital for comprehending projective modules over rings, vector bundles on compact spaces, and various symmetric monoidal categories. The second category establishes conditions for exact sequences, significant in abelian and exact categories; this concept was initially applied in algebraic geometry. The third category focuses on weak equivalence relations, highlighting chain complexes and the associated structures with cofibrations and weak equivalences, commonly known as 'Waldhausen categories.' Furthermore, four essential methodologies for higher algebraic K-theory are recognized: the +–construction (concentrating on rings), group completion techniques (focused on symmetric monoidal categories), Quillen's Q-construction (related to exact categories), and Waldhausen's wS-construction (tailored for categories involving cofibrations and weak equivalences). While these techniques yield the same

K-theory for any ring, each is fine-tuned for a particular case.

Mirror Symmetry

Mirror symmetry began in the field of theoretical physics and has become an essential aspect of algebraic geometry. Key figures in this area include researchers like Candelas, Lynker, Schimmrigk, and Green, who formulated a conjecture that connects pairs of Calabi-Yau manifolds. This concept, which stems from the principles of string theory, has led to extensive mathematical exploration (Andrey, 2022).
The concept of mirror symmetry dates to the mid-1980s, during which it was identified that a string moving around a circle with a radius of 1/R relates to T-duality, an idea that plays a crucial role in mirror symmetry. A landmark paper from 1985 by Philip Candelas, Gary Horowitz, Andrew Strominger, and Edward Witten demonstrated that compactifying string theory on a Calabi-Yau manifold could create a theoretical backdrop like the standard model of particle physics, integrating aspects of supersymmetry. This groundbreaking finding sparked significant interest within the physics community regarding the possibility of Calabi-Yau compactifications as a viable path for developing realistic models of particle physics grounded in string theory. Cumrun Vafa and his team elucidated that it's unlikely to connect a single Calabi-Yau manifold to every physical model. Rather, numerous Calabi-Yau manifolds can produce equivalent physical results (Aganagic, *et al*., 2016). Brian Greene and Ronen Plesser highlighted critical instances of the bonds represented by mirror symmetry by examining the ties between Calabi-Yau manifolds and specific conformal field theories, referred to as Gepner models. Further affirmation of this mirror correspondence came from extensive computational studies by Philip Candelas, Monika Lynker, and Rolf Schimmrigk, which confirmed the frequent pairing of Calabi-Yau manifolds (Fu, et al, 2024). By the early 1990s, the importance of mirror symmetry began to be recognized by mathematicians as physicists like Philip Candelas, Xenia de la Ossa, Paul Green, and Linda Parkes illustrated its ability to address longstanding challenges in enumerative geometry. These advancements were discussed at a conference at the Mathematical Sciences Research Institute (MSRI) in Berkeley, California, in May 1991. During this meeting, inconsistencies were observed between the counts of rational curves provided by Candelas and those calculated by Norwegian mathematicians Geir Ellingsrud and Stein Arild Strømme, who used more rigorous methods. Initially, some mathematicians were skeptical of Candelas' results due to notable discrepancies in their mathematical approaches. However, upon revisiting their calculations, Ellingsrud and Strømme identified an error in their coding; once corrected, their findings corresponded with those of Candelas and his associates (Fu, et al, 2024).
Further developments in mirror symmetry theory include the work of *Kontsevich,* who introduced homological mirror symmetry, providing a categorical framework for understanding mirror symmetry phenomena. This framework has led to numerous conjectures and results connecting algebraic geometry, symplectic geometry and representation theory (Fu, et al, 2024).

In 1990, Edward Witten introduced topological string theory, providing a more approachable interpretation of string theory and highlighting the importance of mirror symmetry within it. This idea has grown to serve as a mathematical representation of mirror symmetry. At the 1994 international congress for mathematicians, Maxim Kontsevich put forth a groundbreaking mathematical concept based on the ideas of mirror symmetry in topological string theory, which he named homological mirror symmetry. This theory illustrates mirror symmetry as a relationship between two mathematical entities: the derived category of coherent sheaves on a Calabi-Yau manifold and the Fukaya category corresponding to its dual structure (Clader, et al, 2018). Around 1995, Kontsevich revisited a previous notion by Candelas regarding a universal method to count rational curves on a quintic threefold, refining it into a definitive mathematical conjecture. By 1996, Alexander Givental published a paper claiming he had resolved Kontsevich's conjecture. Initially, many mathematicians struggled to comprehend Givental's findings, resulting in disputes about their validity. While there was some disagreement over who first published the proof, these papers are now widely seen as validation of the predictions made by physicists about mirror symmetry. In 2000, Kentaro Hori and Cumrun Vafa provided additional physical proofs legitimizing mirror symmetry through T-duality (Clader, et al, 2018). The exploration of mirror symmetry remains lively and fruitful, with significant advancements in studying strings on bounded surfaces. Additionally, mirror symmetry connects with various intriguing research fields, including McKay correspondence, topological quantum field theory, and stability conditions. Despite these advancements, mathematicians still encounter substantial challenges in constructing examples of Calabi-Yau pairs, although progress has been made in this realm (Fu, et al, 2024).

The study of K-theory for p-adic groups

The idea of p-adic numbers was first proposed over a century ago by Kurt Hensel, who built upon the work of Ernst Kummer. The creation of p-adic numbers was mainly driven by the desire to broaden the understanding of power series in number theory. Hensel aimed to connect p-adics with rational numbers, establishing a novel method for approaching power series more generally (Kaplan, 2023).

The groundbreaking algorithms he formulated marked the inception of a new domain concerning p-adic numbers, denoted as Qp. The inherent potential of this newly established numerical structure was quickly recognized, leading to an expansion of its computational applications. Hensel's analogies highlighted the relationships that motivated the development of p-adics, noting the direct resemblances between the established frameworks of real numbers and complex polynomials with p-adic numbers. The similarities recognized by Hensel facilitated the establishment of operations on p-adic numbers that mirrored those in real arithmetic, where various techniques were examined based on fundamental operations like addition and finding inverses (Abe, et al, 2019).

The K-theory of P-adic group has its roots in algebraic topology and number theory, thus offering a powerful tool for studying the algebraic structures underlying P-adic spaces. Adam laid the groundwork for this area by introducing the concept of K-groups and establishing fundamental results such as Bott periodicity theorem (Uttenthal, 2023).Subsequent contributions by *Milnor* and *Quillen* further developed the theory of K-theory, providing deep insight into the algebraic and topological properties of P-adic groups and their associated spaces. The application of K-theory to number theory particularly in the study of Galois representation and arithmetic geometry has led to significant advances in these areas (Vigneras, 2022). In number theory, the p-adic numbers form an extension of the rational numbers when considering a prime number p that is separate from real numbers yet shares some analogous characteristics. These numbers can be expressed in a fashion akin to (potentially infinite) decimal representations but utilize a prime number p as the base instead of ten, and their expansion occurs to the left rather than the right. Although upon reflection, it can be noted that some prior work by Ernst Kummer may have implicitly employed p-adic numbers (Kaplan, 2023). Steinitz published a groundbreaking paper on the abstract theory of fields, which might be regarded as the first work in “abstract algebra.” The p-adic numbers are referenced as a significant impetus for this new theoretical framework. Insights from topology, particularly those from Frechet and Riesz, became more comprehensible, paving the way for an enhanced understanding of p-adic numbers. Hensel demonstrated, through various papers and his writings, that utilizing his p-adic methods provides a more straightforward explanation of divisibility theory within algebraic number fields. This approach seems to have been gradually recognized. An example of this is found in "A New Development of the Theory of Algebraic Numbers" (Andrey, 2022).

Following Steinitz's work, p-adic numbers began to emerge as a standard example in academic papers related to algebra.

The intersection of mirror symmetry and K-theory

The intersection points of mirror symmetry with K-theory within P-adic groups represent a rich area of research that has generated increasing attention in recent years. Building on the work of physicist and mathematicians, several researchers have explored connections between mirror symmetry phenomena and algebraic structures encoded in k-groups (Quillen, 2006).

Algebraic k-theory is a mathematical discipline that links multiple areas including geometry, topology, ring theory, and number theory. Within this field, entities from geometry, algebra, and arithmetic are connected to structures referred to as k-groups, which are groups found in abstract algebra.

K-groups hold crucial information about fundamental objects, yet their computation can be notably intricate. A key unresolved issue is determining the k-groups of integers (Andrey, 2022). K-theory was initially introduced by Alexander Grothendieck in the late 1950s while he was examining intersection theory in algebraic varieties. Grothendieck initially concentrated on the zeroth k-group, denoted as ko, which holds significance across various fields and is essential for the Grothendieck-Riemann-Roch theorem. The exploration of intersection theory remains vital for advancing (higher) algebraic k-theory, especially due to its connections with motivic cohomology and Chow groups. This research also tackles traditional questions in number theory, including quadratic reciprocity and the embedding of number fields within the real number line, while also addressing contemporary issues such as calculating higher regulators and determining values of L-functions (Paranjape, 2018).

The recognition of the lower k-groups initiated a new phase, uncovering fascinating connections between these groups and various algebraic structures. For instance, it has been demonstrated that for any field F, ko(F) is isomorphic to the integers, Z, highlighting a significant relationship with the dimensions of vector spaces. In the context of a commutative ring R, the group ko(R) is linked to the Picard group of R, thereby broadening the conventional class group when R is viewed as the integral domain associated with a number field. Additionally, the group K1(R) corresponds to the group of units, while K2(F) for a number field F is tied to class field theory, the Hilbert symbol, and finding solutions to quadratic equations in completions.

However, Daniel Quillen encountered obstacles when attempting to establish appropriate higher k-groups for rings, leaving numerous fundamental questions about the higher k-groups of algebraic varieties unanswered until Robert Thomason made notable advancements (Clader, et al, 2018).

For instance, *Beilinson* and *Drinfeld* formulated a geometric Langlands corresponds using idea from mirror symmetry and K-theory, providing a unique perspective concerning the relationship between representation theory and algebraic geometry (Paranjape, 2018).

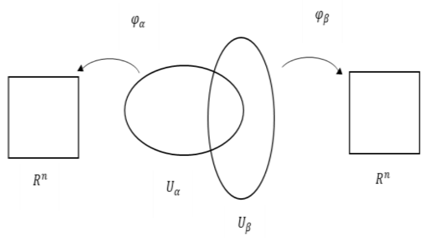
In similar vein, recent work by *Gukov*, *Kapustin* and *Witten* have explored the role of mirror symmetry in the context of modular forms and automorphic representations, drawing connections to the K-theoritic properties of P-adic group (Zelenov, 2023).

In conclusion the literature on mirror symmetry and K-theory of p-adic group reflect a rich tapestry of ideas and techniques from diverse areas of mathematics. While much progress has been made in understanding each of these fields independently, ongoing research continues to uncover deep connections and unexpected phenomena and their intersection, promising further insights into the nature of mathematical reality (Zelenov, 2023).

METHODOLOGY

Manifold

The definition of a manifold involves its coordinate charts. Let us represent the open cover of the topological space M with the set $\{U\alpha\}$. Based on this information, we identify M as an n-dimensional manifold. The coordinate chart is represented by $\varphi\alpha : U\alpha \to R^n$, where the coordinates $x\alpha = (x\alpha)i$, with i ranging from 1 to n, represent the points that correspond to the mapping φα's pre-images. In the area where Uα and Uβ overlap, we can connect the coordinates (xα) and (xβ) using the equation $x\alpha = \varphi\alpha \circ \varphi^{-1}\beta(x\beta)$.



Consider two open sets, Uα and Uβ, each linked to their own coordinate charts, φα and φβ. The transition function gαβ is given by gαβ = φα ◦ φ−1 β plays a critical role in relating these charts. Notably, gαβ is the inverse of gβα, and the composition of the transition functions satisfies the relation gαβgβγgγα = 1. One might also define a manifold focus exclusively on the information relevant to the patches and transition functions that conform to these relationships. A manifold is designated as 'differentiable' if the transition functions demonstrate differentiability, while it is considered 'smooth' if these functions are infinitely differentiable (denoted as C∞). Specifically, when n = 2k, a manifold is labeled "complex" if it features an atlas φα : Uα → Ck, where the transition functions are holomorphic. It is essential to acknowledge that this additional structure might introduce specific constraints. Two complex manifolds can be diffeomorphic if they are examined as real manifolds—indicating the existence of invertible, surjective, differentiable maps linking them—there may be a lack of complex analytic mappings between these manifolds, suggesting they could have distinct complex structures. Similarly, two manifolds might be homeomorphic but still possess different differentiable structures, since differentiability depends on the mappings established within the given context.

Vector Bundles

The construction of vector bundles follows a similar approach, but now each point is associated with an additional element called "fiber," which is a vector space. By maintaining the information related to each point while disregarding the specific connection to the base manifold can be formed through various aspects of the vector space, with a focus on smooth vector bundles (Abe, et al, 2017).

As mentioned earlier, a fundamental vector bundle, referred to as E, can be depicted as the Cartesian product space E = M × V, where M represents the manifold and V denotes an r-dimensional vector space (Givental, et al, 2003). This structure verifies that E is a rank r vector bundle, defined by its projection map π: E → M, which is given by π((m, v)) = m, thereby classifying it as a 'trivial' vector bundle. In local settings, each vector bundle is treated as trivial and can be represented as product spaces. A rank r vector bundle E is viewed as a smooth manifold with a corresponding projection map π: E → M that ties it to the base manifold M. For any point x in M, there exists a neighborhood Ux surrounding x such that the preimage π^{-1}(Ux) is isomorphic to Ux × Rr. The isomorphism ρα: π^{-1}(Uα) ≅ Uα × Rr indicates the choice of coordinates and suggests that the geometric structures may differ due to different "local trivializations." By representing ρα as (π, ψα), we define ψα: π^{-1}(Uα) → Rr. Similarly to how manifolds relate, vectors are linked via sβα = ψβ ◦ ψ^{-1} α, which establishes a connection between (m, vα) in Uα × Rr and (m, ψβ ◦ ψ^{-1} α (vα)). It is crucial to understand that the function ψβ ◦ ψ^{-1} α must act linearly on the fibers Rr, satisfying the conditions sαβ = s^{-1}βα and sαβsβγsγα = 1. Moreover, this information can be employed to assemble the vector bundle using a gluing technique.

$$E = \coprod_\alpha U_\alpha \times \mathbb{R}^r/\sim, \quad (x, v_\alpha) \sim \left(x, s_{\alpha\beta}(x)\right)\left(v_\beta\right)$$

A "section" vα over Uα is defined as a mapping vα: Uα → that functions similarly to a vector-valued function. This signifies that for every point in the base space; there exists a unique point in the corresponding fiber. When two sections are established over regions Uα and Uβ, they can be integrated into a section over the unified region Uα ∪ Uβ, provided they align appropriately in the overlapping area Uα ∩ Uβ, which can be expressed as vβ = sβα(vα).

A "global section" refers to a mapping v: M → E such that the composition π ◦ v acts as the identity on M, indicating that ν serves as a section over Uα. The set of sections of E over a subset U ⊂ M is represented by Γ(U, E), while Γ(E) denotes the entire set of global sections. It is important to understand that sections can be multiplied by functions, meaning that the section's value at any given point is determined by the value of the function at that same point, which allows Γ(U, E) to function as a C∞(U) module. For example, a section of a trivial bundle M × V can be interpreted as a function f: M → V that assigns values in V. A complex vector bundle is characterized as a locally trivial arrangement of complex vector spaces, with its rank being half that of its corresponding real vector bundle. A bundle on a complex manifold is termed "holomorphic" when all transition functions between fibers are holomorphic.

$$\frac{\partial}{\partial x^{a}} = \frac{\partial \widetilde{x^{k}}}{\partial x^{a}} \frac{\partial}{\partial \widetilde{x^{k}}}$$

So

$$s_{a}^{k} = \frac{\partial \widetilde{x^{k}}}{\partial x^{a}}$$

In our equation, we use matrix notation to symbolize s. A vector field is observed on a broad scale, and it is important to acknowledge that every vector bundle inherently includes the zero section among its sections. However, the difficulty of identifying non-vanishing sections becomes quite significant, especially when considering holomorphic structures.

Additional Bundles

We have created a new bundle that makes use of dual vector spaces, deriving its transition functions from the original bundle using methods based on linear algebra.

Similarly, we can generate further bundles by employing dual spaces and tensor products. For example, beginning with the vector bundle E, we can create the vector bundle E ⨁ E, where the fiber at each point x is represented as Ex ⨁ Ex. If we denote the transition functions for E as sαβ, then the transition function for E ⨁ E can be formulated as sαβ ⨁ sαβ. Additionally, by examining two vector bundles E and F over the manifold M, we can derive new bundles such as E ⨁ F, Hom(E,F), E*, and E ⨂ F, among others. Importantly, the bundle E ⨂ E can be separated into two parts: (E ⨂s E) ⨁ (E ⨂a E), with 's' indicating the symmetric part and 'a' indicating the anti-symmetric part. It's important to highlight that for a vector space V, the anti-symmetric tensor product Λ2V—also known as V ∧ V or V ⨂a V—is derived from the quotient V ⨂ V/I, where I is the subspace spanned by elements of the form vi ⨂ vj + vj ⨂ vi. The equivalence class [vi ⨂ vj] is represented as vi ∧ vj, leading to the relation vi ∧ vj = -vj ∧ vi. Thus, we define E ⨂a E as Λ2E, and this concept also applies to ΛpE. When we consider two bundles E and F over the manifold M, a map f: E → F is identified as a bundle map if it establishes a relationship between the total spaces of these bundles, maintains linearity across the fibers, and properly aligns with the projections of the respective bundles. In this context, we define the kernel as Ker(f) ⊂ E, and the cokernel as Coker(f) = F/Im(f), clarifying the interpretations of the fibers according to linear algebra concepts. The bundles ΛpT∗M are particularly significant as they represent completely anti-symmetric p-multi-linear mappings (or p-tensors) applied to tangent vectors. For an n-dimensional manifold M, the component ΛpT∗M denotes a rank bundle of anti-symmetric p-tensors, often called "p-forms." The sections of ΛpT∗M are usually referred to as Ωp(M). It is also crucial to grasp that for any vector space V, we have Λ0V = 0, indicating that Ω0(M) consists of sections of the trivial line bundle that correspond to functions.

Metrics on Manifolds

In a vector space, the inner product is crucial for understanding the lengths of vectors and the angles between them. When examining a manifold, the tangent vector spaces related to the fibers of the tangent bundle resemble the tangent planes found on the surface of a sphere in three-dimensional space. Consequently, the inner product is modified to fit this context. This inner product is known as a "metric," denoted as g, which captures the essential aspect of measurement within geometric contexts. Thus, for vectors v and w located at a particular point x, g(v,w) produces a real number. It is essential to note that both g and the vectors v and w rely on x, particularly when v(x) and w(x) are vector fields. The metric g must exhibit bilinearity over the fibers and demonstrate symmetry, adhering to certain conditions: g(v, w) = g(w, v), g(λv, w) = λg(v, w) = g(v, λw). In an appropriate coordinate chart, this relationship can be expressed as follows:

$$v = v^a \frac{\partial}{\partial x^a},$$

so

$$g(v, w) = g\left(v^a \frac{\partial}{\partial x^a}, w^b \frac{\partial}{\partial x^b}\right) = v^a w^b g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right).$$

We define

$$g_{ab} \equiv g\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right),$$

In a local region, the function g is described by its matrix components (x), and a manifold that is endowed with a positive-definite metric at all points is referred to as a "Riemannian manifold."

Heck Algebra and P-Adic Groups

Let G be a totally disconnected group, defined as a locally compact, second-countable, and Hausdorff topological group. The identity element e has a countable basis of neighborhoods composed of compact open subgroups. Let R represent a unital ring that may not be necessarily commutative and can include the rational numbers Q. The Hecke algebra H(G; R) associated with G over R consists of locally constant functions with compact support that take values in R, with the multiplication defined through convolution using a Q-valued left-invariant Haar measure on G. Extended forms of the Hecke algebras, referred to as H(G; R, ρ, ω), include twists ρ due to a G action on R and involve a central character ω. Significantly, Hecke algebras may not universally be unital. A module M over the Hecke algebra is considered non-degenerate if it meets the condition H(G; R)·M = M. Furthermore, a representation of G on an R-module V is termed smooth if the isotropy group of G's action on V constitutes an open set. The category of non-degenerate H(G; R)-modules aligns with the category of smooth representations on R-modules. A reductive p-adic group is formulated as the points over F of an algebraic group over F, distinguished by its reductive identity component, where F represents a non-Archimedean local field, which can either be a finite extension of p-adic numbers or the field of formal Laurent series k((t)) over a finite field k. Importantly, reductive p-adic groups are identified as td-groups. Moreover, a complex Bruhat-Tits building is associated with a reductive p-adic group, highlighting a cocompact proper isometric action by G. This building can also be described as a simplicial complex that accommodates simplicial and smooth actions of X.

The Calabi-Yau Criterion

We now focus on the Calabi-Yau criterion regarding the trivial nature of the canonical bundle. The canonical bundle represents the line bundle connected with the determinant of the highest antisymmetric tensor product of the holomorphic cotangent bundle. Consequently, its first Chern class is the negative of the first Chern class of the holomorphic tangent bundle, TM. Therefore, the condition for the canonical bundle to be considered trivial can be simply stated as (TM) = 0. By examining the relationship between the first Chern class and the curvature associated with a connection, we conclude that the class of R is trivial from a cohomological viewpoint.

Construction of Calabi-Yau Manifolds

A Calabi-Yau manifold in algebraic geometry is defined as a complex projective manifold V that possesses a trivial canonical sheaf, meaning it is isomorphic to the structure sheaf, and its cohomology groups vanish for all p values between 1 and n-1, where n indicates the dimension of V. It is commonly assumed that V's fundamental group is finite to prevent certain unusual cases. From the standpoint of differential geometry, these manifolds are classified as Kähler manifolds that are characterized by having zero Ricci curvature and a specific type of holonomy group. An important classification theorem states that for any compact Kähler manifold X possessing a zero first Chern class, there exists a non-ramified finite cover from a certain manifold to X. This implies that the structure can be expressed as a combination of a complex torus T, a simply connected Calabi-Yau manifold, and a symplectic manifold which includes a non-degenerate holomorphic 2-form at each point.

Theorem 1 (Yau): If X is a Calabi-Yau manifold with a Kähler metric g and a Kähler form ω ε, then there exists a unique Kähler metric gy characterized by zero Ricci curvature, referred to as the Yau metric, such that the respective form ωy is equal to ϵ.

The P-adic Numbers and Hensel's Lemma Kurt

Hensel is widely recognized for his in-depth historical analysis of p-adic numbers, having introduced what is known as Hensel's analogy in the early 1900s. He connected the well-established area of complex polynomials with the emerging notion of p-adic numbers. The integer set $\mathbb{Z}$ and the rational number set $\mathbb{Q}$ are closely linked in a way that every number can be represented as a fraction of integers, forming significant connections. Similarly, complex polynomials related to the ring $\mathbb{C}$ demonstrate a relationship with their field of quotients, illustrated via a specific function where $\mathbb{C}$ represents the field of quotients. The quotient ring associated with complex polynomials is identified as R = R. Theorem 3.1 validates the existence of this quotient ring.

Resolution of Smooth Representations

Let G denote a reductive p-adic group. In this context, the category of smooth complex representations is noetherian and possesses finite cohomological dimensions. As a result, any finitely generated smooth complex representation can be represented as a finite resolution of the form Pn → Pn−1 → · · · → P0 → V, where each Pi is finitely generated and projective. A smooth G-representation is considered admissible if the U-fixed vector subspace VU is finite-dimensional for every compact open subgroup U of G. It is termed compactly induced if it arises from the compact induction of a finitely generated projective U-representation corresponding to a specific compact open subgroup U of G. Schneider and Stuhler demonstrated that for finitely generated admissible representations V, one can judiciously select the Pi in the resolution to be finite direct sums of compactly induced representations.

RESULTS AND DISCUSSION

In this section, we present the results as obtained from the methods outlined above. The results demonstrate how 3D mirror symmetry was utilized to develop a group of polynomials with integer coefficients, which are intended to solve the quantum differential equation modulo a certain integer. The sequence was shown to converge in the p-adic norm to Okounkov's vertex function, which is a prime number, and confirmed that it meets the Dwork-type congruences. The vertex functions serve as analogs to Givental's J-functions within the realm of quantum cohomology. These vertex functions are expressed in the form of power series.

$$V(z) = \sum_{d=0}^{\infty} c_d z^d \in \mathbb{Q}[\![z]\!]$$

The coefficients indicate how many degree d-rational curves exist within the quiver variety X. These figures are obtained from the regularized integral of the virtual fundamental class

$$c_d := \int_{QM_d(X,\infty)} \omega^{vir}$$

This part begins a study of the arithmetic characteristics of the moduli space of Quasimaps of degree d that come from a rational curve mapping into X, which has specific properties at certain points. To achieve this, we examine the vertex function associated with the most basic Nakajima quiver variety, which is represented by the cotangent bundle over the Grassmannian. For a prime number p, we establish an initial term and then generate a series of polynomials that tend toward the vertex function.

$$\lim_{s\to\infty} \mathrm{T}_s(z) = V(z).$$

Within the framework of the P-adic norm, convergence is established, leading us to refer to the polynomials as p-adic approximations. We notice that, in contrast to vertex functions, these p-adic approximations exhibit a range of exceptional correspondences.

Theorem 2: The p-adic approximations $\mathrm{T}_s(z) \in \mathbb{Z}[z]$, satisfy the Dwork-type congruences:

$$\frac{\mathrm{T}_{s+1}(z)}{\mathrm{T}_s(z^p)} = \frac{T_s(z)}{T_{s-1}(z^p)} \bmod p^s \qquad 1$$

with $s = 1,2, \dots$

Such congruences were essential in Dwork's studies since they formed the foundation of the theory on p-adic hypergeometric equations. Indeed, by examining theorem 2, we derive estimates for the truncated forms of the hypergeometric function $\mathrm{F}_1\left(\frac{1}{2}, \frac{1}{2}, 1; \mathrm{z}\right)$ considered by Dwork in his main reference, but it differs.

Theorem 3: The vertex function can be presented as an infinite product:

$$V(z) = \prod_{i=0}^{\infty} \frac{T_m\left(z^{p^i}\right)}{T_{m-1}\left(z^{p^{i+1}}\right)} mod\ p^m, m = 1,2 \dots$$

For m = 1, we obtain

$$V(z) = \prod_{i=1}^{\infty} T_1(z^{p^i}) \bmod p.$$

The creation of p-adic approximations is influenced by the p-adic evaluations of hypergeometric solutions related to the KZ equations, alongside the concepts of 3D-mirror symmetry. This research focuses on a particular quiver variety named 3D-mirror X. For each prime number p, we develop a polynomial associated with this quiver, where extra variables represent the Chern roots of the tautological bundle on the quiver variety. This polynomial can be interpreted as a p-adic polynomial approximation tied to the superpotential in 3D quantum field theory along the Higgs branch. Additionally, we will introduce a notable coefficient that is important in relation to this polynomial.

The normalized vertex function associated with a quiver variety is recognized as the only analytic solution to the quantum differential equation that governs the quantum cohomology of that variety. For example, when n=1, 2, etc., these functions correspond to standard generalized hypergeometric equations. Furthermore, it is essential to emphasize that in this context, the relevant polynomial is linked to the Hasse-Witt invariant of an elliptic curve, which was initially identified by Igusa as a solution to the Gauss hypergeometric differential equation modulo p. Moreover, the congruences mentioned in (2) suggest that the sequence behaves like a Cauchy sequence, converging uniformly to an analytic function evaluated at p over a broad range. This function encapsulates the p-adic analytic continuation of the expression as a ratio of power series around a particular point. A modular transformation identity is valid for points within a specific interval, where the constant d is contingent on the selected parameters.

Vertex functions of $T^*G_r(k,n)$

The vertex function linked to the cotangent bundle defined in the context of a Grassmannian space $T^*G_r(k,n)$ is given by the power series:

$$V(z) = \sum_{d=0}^{\infty} c_d(u_1, \dots, u_n, \hbar) z^d \qquad 2$$

With the coefficients $c_d(u_1, \dots, u_n, h) \in \mathbb{Q}(u_1, \dots, u_n, h, \epsilon)$ given by:

$$\sum_{\substack{d_1,\dots d_k \\ d_1+\dots+d_k=d}} \left(\prod_{i,j=1}^{k} \frac{(\epsilon - u_i + u_j)_{d_i - d_j}}{(h - u_i + u_j)_{d_i - d_j}}\right)\left(\prod_{j=1}^{n} \quad \prod_{i=1}^{k} \frac{(h + u_j - u_i)_{d_i}}{(\epsilon + u_j - u_i)_{d_i}}\right) \qquad 3$$

Where $(x)_d$ denote the Pochhammer symbol with step ϵ:

$$(x)_d = \begin{cases} x(x+\epsilon)\dots(x+(d-1)\epsilon), & d > 0 \\ 1, & d = 0 \\ \frac{1}{(x-\epsilon)(x-2\epsilon)\dots(x+d\epsilon)} & d < 0 \end{cases} \qquad 4$$

In this sequence, the coefficient of degree d is closely connected to the quantity of rational curves of degree d within X. This calculation is carried out in a more accurate manner with the assistance of the equivariant integral.

$$c_d(c,\hbar) = \int_{[QM_d(X,\infty)]^{vir}} \omega^{vir}$$

Regarding the class of virtual fundamentals for the moduli space concerning quasimaps originating from a defined space and mapping to X, which targets a specific fixed point under the action of the torus in X. In this scenario, the parameters involved pertain to the equivariant parameters associated with the torus, which exerts its influence on the moduli space as described below:

• The torus exerts a natural action on the moduli space by scaling the coordinates with designated weights. This results in an induced torus action regarding the moduli space, which ultimately manifests as a T action.

• The torus modifies the cotangent fibers in X with a defined weight h, leading to a corresponding action of T on the relevant space.

• The actions stem from the origins of the quasimaps that stabilize the points at both 0 and infinity. The variable represents the weight linked to the tangent space.

With k = 1 and n = 2, concerning the cotangent bundle over the projective space, the vertex function indicated in (3) is associated with the Gauss hypergeometric function is

$$V(z) = 2^{F_1\left(\frac{\hbar}{\epsilon},\frac{u_2-u_1+h}{\epsilon};\frac{u_2-u_1+\epsilon}{\epsilon};z\right)}.$$

$$V(z) = \int_\gamma \Phi(x,z)dx$$

We will now provide a precise mathematical description of the construction for a properly chosen real-dimensional X-dimensional contour, specifically for this case. Let us consider the assumption that we pair the dimensions (k, n) with a framed quiver. This

$$v_i = \begin{cases} i & i < k \\ k & k \le i \le n-k \\ n-i & n-k < i \end{cases}$$

In this study, we consistently analyze the vertex function (4.2) focusing on a specific set of equivariant parameters

$$u_1 = \cdots u_n = 0, \qquad \epsilon/\hbar = \omega, \qquad \omega \in \mathbb{C}.$$

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Subsequently, we adjust it to be a rational number equal to 0. Following this illustration, when k equals 1 and n equals 2, the specific vertex function is expressed in the following manner.

$$V(z) = 2^{F_1}(\omega,\omega;1;z) = \sum_{d=0}^{\infty} \binom{\omega}{m}^2 z^d$$

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If $\omega = \frac{1}{2}$, then the initial the coefficients in this power series are limited in number.

$$V(z) = 1 + \frac{1}{4}z + \frac{9}{64}z^2 + \frac{25}{256}z^3 + \frac{1225}{16384}z^4 + \mathcal{O}(z^5).$$

3D mirror symmetry and integral representations of cohomological vertex functions

The concept of 3-dimensional mirror symmetry offers integral representations for the vertex functions. This symmetry connects a symplectic variety X with a corresponding 3-dimensional mirror variety, along with a function known as the superpotential of that variety. A key prediction motivated by 3D mirror symmetry indicates that the functions related to vertices can consequently be expressed as follows.

quiver features complex one-dimensional framings that are exclusively positioned at the vertices k and n-k. The dimension vector is computed using the formula presented here:

Let us explore Nakajima's quiver variety concerning these parameters. It has been demonstrated that this variety serves as a three-dimensional mirror to the specified entity, suggesting that the vertex functions of both varieties match perfectly.

For a particular vertex i assigned a dimension in a quiver, we define a set of variables. In algebraic topology, these variables are interpreted as the Chern roots linked to the i-th tautological bundle relevant to the corresponding quiver variety. For a framed vertex of a certain dimension, we create a set of variables indexed by j=1. Thus, we developed the super potential of a quiver variety: To depict an arrow starting from vertex j and ending at vertex i, we incorporate a distinct factor

$$\prod_{a=1}^{v_i}\prod_{b=1}^{v_j}\left(x_{i,a}-x_{j,b}\right)^{-\omega}$$

In the quiver, a vertex m corresponds to a factor

$\prod_{1\le i<v_m}\left(x_{m,i}-x_{m,j}\right)^{2\omega}$ and $\left(\prod_{j=1}^{v_m}x_{m,j}\right)^{-1+\omega}$

The quiver that depicts the mirror variety gives rise to the following super potentials based on these principles:

$$d_j^{(i)}=j$$

For values of i ranging from 1 to n-1 and j from 1 onward, the primary entities of this study are the subsequent polynomials. We define a polynomial according to the formula

$$T_S(z):=(-1)^{\frac{(P^S-1)r}{}N}coeff_{x^{dp^s}-1}\left(\Phi_s(x,z)\right)$$

Where $coeff_{x^{dp^s}-1}$ denotes the coefficient of the nominal

$$x^{dp^s}-1=\prod_{i=1}^{n-1}\prod_{j=1}^{v_i}x_{i,j}^{jp^s-1}$$

in the polynomial $\Phi_s(x,z)$. N represents the value from theorem (4)

$$\Phi(x,z)=\left(\prod_{i=1}^{n-1}\prod_{j=1}^{v_i}x_{i,j}\right)^{-1+\omega}\left(\prod_{m=1}^{v_m}\prod_{1\le i<j\le v_m}\left(x_{m,j}-x_{m,i}\right)\right)^{2\omega}\times\prod_{i=1}^{n-2}\prod_{a=1}^{v_i}\prod_{b=1}^{v_{i+1}}\left(x_{i,a}-x_{i+1,b}\right)^{-\omega}\times\left(\prod_{i=1}^{k}\left(z_{k,1}-x_{k,i}\right)\left(z_{n-k,1}-x_{n-k,i}\right)\right)^{-\omega}$$

This method draws attention to the critical role of super potentials as core functions in the theoretical landscape of integral representations pertaining to the trigonometric Knizhnik-Zamolodchikov equations.

Theorem 4: The function pertaining to the vertex of $X=T^*Gr(k,n)$ with symmetrical properties adjusted at (4) focuses on approximating the vertex function using p-adic methods.

We will establish a degree vector d by using the following definition:

$$V(z)=\sum_{m=0}^{\infty}c_m z^m,\ \ T_s(Z)=\sum_{m=0}^{degT_s(z)}c_{s,m}z^m.$$

For each m, the series of integers converges with respect to the p-adic norm

$$\lim_{s\to\infty}c_{s,m}=\ c_m.$$

Example 5: Let k=1, n=2, $\omega=1/2$. In this case $T_s(z)$ is the coefficient of $x^{p^{s-1}}$ in the polynomial

$$\Phi_s(x,z)=-1^{\frac{p^s-1}{2}}\left(x(x-1)(x-z)\right)^{\frac{p^s-1}{2}}$$

An elementary computation gives

$$T_s(z)=\sum_{d=0}^{p^s-1}\binom{\frac{p^s-1}{2}}{d}^2 z^2.$$

In the realm of p-adic norms, we can state that

$$\lim_{s\to\infty}\left(\frac{\frac{p^s-1}{2}}{d}\right)=\left(\frac{\frac{-1}{2}}{d}\right),$$

In the limit as s approaches a particular value, we derive the hypergeometric series (4.6). with $\omega = 1/2$.

The *p*-adic number and hensel's lemma

In this section, we shall refer to $\mathbb{C}(X)$ as field of quotients. Furthermore, for any prime number p, it is possible to express any positive integer m as

$$m = a_0 + a_1 p + \cdots a_n p^n = \sum_{i=0}^{n} a_i p^i$$

For some non-negative values and a sequence with components from the defined set, we will provide a preliminary comparison between standard integers and p-adic numbers as follows: Every integer can be represented in the standard base of ten, which can further be broken down into components.

$$327 = 7 \times 10^0 + 2 \times 10^1 + 3 \times 10^2.$$

It's important to note that this element does not possess an inverse in $\mathbb{Z}$! By defining p as 7 and looking into the 7-adic expansion of 327, specifically

$$327 = 5 \times 7^0 + 4 \times 7^1 + 6 \times 7^2$$

Subsequently, we will illustrate that we can find its inverse within the 7-adic numbers.

$$f(X) = a_0 + a_1 (X-\propto) + \cdots + a_n(X-\propto)^n = \sum_{i=0}^{n} a_i (X-\propto)^i,$$

The degree of the quantity is an arbitrary integer, and there exists a sequence generated from Taylor's formula, which we will not concentrate on; our main point is that such a representation is always achievable.

$$f(X) \rightarrow expansion\ around \qquad (X-\propto)$$

defines the integration of fields $\mathbb{C}(X) \rightarrow \mathbb{C}(X-\propto)$, where $\mathbb{C}((X-\propto))$ represent all expansions:

$$f(X) = \frac{p(X)}{q(X)} = an_0(X-\propto)^{n_0} + ac + 1(X-\propto)^{n_0+1} + \cdots = \sum_{i\geq n_0}(X-\propto)^i$$

Correspondingly, any positive rational number $x = \frac{c}{d}$ can be written in the structure of a formal power series

$$x = \sum_{n\geq n_0} a_n\, p^n \ where\ x = pc\frac{\bar{c}}{\bar{d}}\ with\ p$$ does not divide $\bar{c}\bar{d}$

Without the typical notation for convergence related to the usual absolute value, this illustrates the idea of multiplicity.

Example 6: Consider a 5-adic extension of $\frac{1}{2}$. First we divide $\frac{1}{2}$ by 5 as follows

$$\frac{1}{2} = -\frac{1}{2}.5 + 3,$$

So the last digit is 3. Now taking $-\frac{1}{2}$ and dividing by 5 we get

$$-\frac{1}{2} = -\frac{1}{2}.5 + 2,$$

Since the new quotient is now equal to $-\frac{1}{2}$, we obtain 2 in the expansion forever, yielding

$$\frac{1}{2} = (\ldots 2223)_5$$

The p-adic digit, p-adic whole number, and p-adic value

In this part, we will move away from the formal construction of p-adic numbers towards a more intuitive understanding. To bridge the theory with practical calculations, we will define various concepts in the context of p-adics and develop algorithms for addition and multiplication with these objects. We will also examine some fundamental properties associated with them.

As is the case with the decimal system (base 10), a number will qualify as a p-adic integer if it belongs to the p-adic number system expansion consists solely of non-negative powers of p. However, there is a deeper understanding to uncover, so we will elaborate on p-adic integers in more detail below.

A digit in p-adic representation is identified as a natural number d that satisfies the condition 0 ≤ d < p, with p being a prime number.

$$\sum_{i=0}^{\infty} d_i p^i = \cdots d_i \dots\ d_1 d_0 \mathbb{Z}$$

We define $\mathbb{Z}_p$ as the collection of p-adic integers:

$\mathbb{Z}_p$: = $\{(d_i)_{i \epsilon \mathbb{N}} :\ d_i\ is\ a\ p - adic\ digit\}$ = $\{\dots d_i \dots\ d_1 d_0 : d_i\ is\ a\ p - adic\}$ that relate to the *p*-adic number such that

$$x \,\epsilon\, \mathbb{Z} \Leftrightarrow\ x =\ \sum_{n \geq 0} a_n p^n.$$

Example 7: To determine the 5-adic expansion of the integer 583, we proceed with the calculations.

$$583 = 3 \times 5^0 + 1 \times 5^1 + 3 \times 5^2 + 4 \times 5^3,$$

What translates into (583)10 = (4313)5 according to the formulation mentioned above? It's important to note that, much like standard integers, the p-adic integers only incorporate non-negative powers of p in their expansions, meaning the lowest power present must be 0. Clearly, $\mathbb{Z}$p is not a field, as, similar to $\mathbb{Z}$, it lacks inverses for some of its elements. Therefore, we aim to demonstrate that excluding this condition, the set Zp satisfies all other axioms and consequently qualifies as an integral domain.

Dwork type congruences

Theorem 8: The Dwork-type congruences are satisfied by the polynomials:

$$\frac{T_{S+1}(z)}{T_S(Z^p)} = \frac{T_S(z)}{T_{S-1}(Z^p)} \text{ mod } p^s$$

With s=1,2,… and $T_0(z) = 1$.

Corollary 9: For *m* = 0,…,*s*-1, the polynomials $T_s(z)$ sastisfy the following set of congruences:

$$T_s(z) = \frac{T_{S-m}(z)T_{S-m}(Z^p)\dots T_{s-m}\left(Z^{p^m}\right)}{T_{S-m-1}(Z^p)\dots T_{S-m-1}\left(Z^{p^m}\right)} \text{ mod } P^{s-m} \quad 7$$

Specifically, for the value m equals s minus one, we arrive at

$$T_s(z) = T_1(z)T_1(z^p) \dots T_1(z^{p^{s-1}}) \text{ mod } p$$

Proof: By applying substitution $s \to s - 1$ we rewrite relation (4.6) in the form

$$T_s(z) = \frac{T_{S-1}(z)T_{S-1}(z^p)}{T_{S-2}(z^p)} \text{ mod } P^{s-1} \quad 8$$

which gives (7) for *m*=1.

For the next phase, we utilize the relationship to swap out the factors in the numerator of (8) with

$$T_{s-1}(z) = \frac{T_{S-2}(z)T_{S-2}(z^p)}{T_{S-3}(z^p)} \text{ mod } P^{s-2}$$

Which gives

$$T_s(z) = \frac{T_{S-2}(z)T_{S-2}(z^p)T_{S-2}\left(z^{p^2}\right)}{T_{S-3}(z^p)T_{S-3}\left(z^{p^2}\right)} \text{ mod } P^{s-2}$$

Continuing with the principle of induction, we arrive at (7) after m steps

Convergence and analytic continuations

***P*-adic discs**

For $u \in \mathbb{F}_p$, let $\tilde{u} \in \mathbb{Z}_p$ be its Teichmuller lift, which is the unique lift that satisfies the given condition$\tilde{u}^p = \tilde{u}$**.** we will denote it as follows:

$$D_u = \{a \in \mathbb{Z}_p : |a - \tilde{u}|_p < 1\}$$

The *p*-adic discs gives a partition

$$\mathbb{Z}_p = \bigcup_{u \in F_p} D_u$$

For a polynomial $B(z) \in \mathbb{Z}_p[z]$, let us define

$$D_B = \{a \in \mathbb{Z}_p : |B(a)|_p = 1\}$$

Let $\tilde{B}$(z) represent the projection of $B(z)$ onto $\mathbb{F}_p[z]$, then D_B is the amalgamation of discs:

$$D_B = \bigcup_{u \in F_{p, B(u) \neq 0}} D_{u.}$$

Given k, we have

$$\left\{a \in \mathbb{Z}_p : \left| B\left(a^{p^k}\right)\right|_p = 1\right\} = D_B.$$

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Limits of uniform convergence

In relation to p-adic evaluations of the vertex function $T_s(z)$ we define

$$\mathfrak{D} = \left\{z \in \mathbb{Z}_p : |T_1(z)|_p\right\} = 1$$

Theorem 10: The sequence of function $I_s(z) = \frac{T_{s+1}(z)}{T_s(z^p)}, s = 0,1,\ldots$ converges uniformly on $\mathfrak{D}$ to an analytic $\mathbb{Z}_p$valued function. If the function is represented by $I(z)$, then for every $a \in \mathfrak{D}$ we have $|I(a)|_p = 1$.

Conclusion

During our study of 3D mirror symmetry, we have successfully developed a polynomial system featuring integer coefficients, which is designed to address the quantum differential equation associated with a prime integer. We intend to show that this sequence converges with Okunkov's vertex function under the p-adic norm framework. Our findings validate that the sequence complies with Dwork-type congruences. In addition, we investigated p-adic numbers and successfully obtained the p-adic representation for various integers.

Conflict of Interest

The authors have declared no conflict of interest regarding the manuscript.

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