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On Exact Discretization Technique for the Solution of Oscillatory Problems of Third Order

Sunday, J.[1], Abdullahi, J.[2], Manga, I.[2] and Sarjiyus, O.[2]
[1]Department of Mathematics, Adamawa State University, Mubi, Nigeria
[2]Department of Computer Science, Adamawa State University, Mubi, Nigeria
Contact: sunday578@adsu.edu.ng

Abstract

One of the techniques needed to construct an efficient method for the solution of oscillatory problems is exact discretization. This technique is adopted in the construction of a new Exact Finite Difference Scheme (EFDS) for the solution of third order oscillatory problems. In carrying out the construction of the method, it was assumed that at any point within the interval of integration, the approximate/numerical solution coincides with the exact/theoretical solution. The analysis of the method was also carried out to show that third order oscillatory problems that possess solutions also have their corresponding EFDS. The method derived was then applied on some modeled third order oscillatory problems and from the results obtained, it is obvious that the EFDS derived did not exhibit any numerical instabilities. As a matter of fact, the computed solutions of the EFDS are exactly equal to the exact solutions.

Keywords: EFDS; exact discretization; NSFDM; oscillation; solutions; third-order

Introduction

According to Sunday (2018), one of the most challenging equations being encountered nowadays are the oscillatory differential equations. This is because their solutions are composed of smooth varying and 'nearly periodic' functions, i.e. they are oscillations whose wave form and period varies slowly with time (relative to the period), and where the solution is sought over a very large number of cycles, (Stetter, 1994). For such problems, one cannot and does not want to follow the trajectories; instead one resort to finding their approximate solutions or the computation of their quasi-envelops. The difficulty of solving such problems is explained by the necessity to ensure correct values of the amplitude and phase angle over many periods.

In this research, exact discretization technique shall be employed in developing an EFDS for the solution of third order oscillatory problems of the form:

$$\frac{d^3y}{dt^3} = f(t, y, y', y'', t, \lambda), \quad y(t_0) = y_0, \; y'(t_0) = y_0', \; y''(t_0) = y_0'', \quad t \in [0, T) \qquad (1)$$

where $f(y, y', y'', t, \lambda)$ is has a unique solution over the interval $0 \le t < T$, $T = \infty$ and for λ in the interval $\lambda_1 \le \lambda \le \lambda_2$. A major advantage of having an exact difference equation model for a differential equation is that questions related to the usual consideration of consistency, stability and convergence need not arise (Mickens, 1994).

It is assumed that equation (1) satisfies the existence and uniqueness theorem of differential equations. It is also assumed that the solutions to equations of the form (1) are bounded. It is important to state that a solution $y(t)$ to equation (1) is said to be bounded if,

$$\sup_{t \in \Re} \|y(t)\| < \infty \qquad (2)$$

Equation (1) has a wide range of applications in engineering, thermodynamics and other real life problems. They are also applied in studying thin-film flows (Duffy and Wilson, 1997), chaotic systems (Genesio, R. and Tesi, 1992), electromagnetic waves (Lee, Fudziah, and Norazak, 2014), among others.

The EFDS is a special form of Non-Standard Finite Difference Method (NSFDM). The exact discretization technique method was first discussed by Potts (1982). He considered the question that whether a linear ordinary difference equation that has the same general solution with the given linear ordinary differential equation can be determined. Also, according to Agarwal (2000), any ordinary differential equation has the exact discretization if its solution exists. More importantly, studies have shown that this statement is also true for partial differential equations (Roeger, 2008).

Some EFDS have been derived by authors to directly solve third order problems of the form (1), Rucker (2003) constructed an EFDS for a nonlinear partial differential equation having linear advection and an odd-cubic reaction term. Mickens, Oyedeji and Rucker (2005) derived an EFDS for second order linear equations. Roeger (2008) derived EFDS for a two-dimension linear system with constant coefficients. Sunday (2010) also developed an EFDS for the numerical solution of initial value problems in ordinary differential equations. Cieslinski (2011) developed an EFDS for classical harmonic oscillator equation. In 2014, Zhang, Wang and Ding also constructed EFDS and NSFDS for Burgers and Burgers-Fisher equations. Also, see the works of Adesanya, Udoh, and Ajileye (2013), Lee, Fudziah, and Norazak (2014), Sunday (2018) for block methods developed for the solution of problems of the form (1).

Definition 1

A differential equation is said to be oscillatory if,

(i) all the nontrivial solution of (1) have an infinite number of zeros (roots) on $x_0 \le x < \infty$, see Kanat (2006), and

(ii) it has at least one oscillating solution, see Borowski and Borwein (2005)

Definition 2: Anguelov and Lubuma (2001)

A finite difference scheme is called non-standard finite difference method, if at least one of the following conditions is met;

i) in the discrete derivative, the traditional denominator is replaced by a non-negative function ϕ such that,

$$\phi(h) = h + o(h^2),\ as\, h \to 0 \quad (3)$$

ii) non-linear terms that occur in the differential equation are approximated in a non-local way i.e. by a suitable functions of several points of the mesh. For example,

$$y^2 \approx y_n y_{n+1}, y_{n-1}y_n,\ y^3 \approx y_{n-1}y_n y_{n+1}, y_n^2 y_{n+1}$$

.

Definition 3: (Mickens, 1994)

An EFDS is one for which the solution to the difference equation has the same general solution as the associated differential equation.

Below, we give the standard finite discrete representations for some derivatives;

$$\frac{dy}{dt} \to \frac{y_{n+1} - y_n}{h} \quad (4)$$

$$\frac{d^2y}{dt^2} \to \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \quad (5)$$

$$\frac{d^3y}{dt^3} \to \frac{y_{n+1} - 3y_n + 3y_{n-1} - y_{n-2}}{h^3} \quad (6)$$

Analysis of the Exact Finite Difference Scheme

In carrying out the analysis of the EFDS, it is important to state that the solution to equation (1) can be written as,

$$y(t) = \phi(\lambda, y_0, y_0', y_0'', t_0, t) \quad (7)$$

with

$$\left.\begin{aligned} \phi(\lambda, y_0, t_0, t_0) &= y_0 \\ \phi(\lambda, y_0', t_0, t_0) &= y_0' \\ \phi(\lambda, y_0'', t_0, t_0) &= y_0'' \end{aligned}\right\} \quad (8)$$

Consider a discrete model of equation (1) given by,

$$y_{n+1} = g(\lambda, h, y_n, y_{n-1}, y_{n-2}, t_n),\ t_n = hn \quad (9)$$

Its solution can be expressed in the form,

$$y_n = \psi(\lambda, h, y_0, t_0, t_n) \quad (10)$$

with

$$\left.\begin{aligned} \psi(\lambda, y_0, t_0, t_0) &= y_0 \\ \psi(\lambda, y_0', t_0, t_0) &= y_0' \\ \psi(\lambda, y_0'', t_0, t_0) &= y_0'' \end{aligned}\right\} \quad (11)$$

Definition 4

Equations (1) and (9) are said to have the same general solution if and only if

$$y_n = y(t_n) \quad (12)$$

for arbitrary values of h.

Theorem 1

The differential equation (1) has an EFDS given by the expression,

$$y_{n+1} = \phi[\lambda, y_n, y_{n-1}, y_{n-2}, t_{n-2}, t_{n-1}, t_n, t_{n+1}] \quad 13)$$

where ϕ is that of equation (7).

Proof

The group property of the solu tions to equation (1) gives,

$$y(t+h) = \phi[\lambda, y(t), t-2h, t-h, t, t+h] \quad (14)$$

If we now make the modifications,

$$t \to t_n,\ \ y(t) \to y_n \quad (15)$$

then, equation (14) becomes,

$$y_{n+1} = \phi[\lambda, y_n, y_{n-1}, y_{n-2}, t_{n-2}, t_{n-1}, t_n, t_{n+1}] \quad (16)$$

This is the required ordinary difference equation that has the same general solution as equation (1). It is important to note the following implications from the theorem above.

(i) If all solutions of (1) exist for all time, $T = \infty$, then equation (14) holds for all $t\ and\ h$. Otherwise, the relation is assumed to hold whenever the right side of (14) is well defined

(ii) The theorem is only an existence theorem. That is, if an ordinary differential equation has a solution, then an EFDS exists. According to Mickens (1994), no guidance is given as to how to actually construct such a scheme.

(iii) A major implication of the theorem is that the solution of the difference equation is exactly equal to the solution of the ordinary differential equation on the computational grid for fixed, but, arbitrary step-size h.

Derivation of the Exact Finite Difference Scheme via Exact Discretization

Theorem 1 stated earlier shall be adopted in constructing an EFDS (via discretization) for third order oscillatory problems of the form (1) for which exact general solutions are explicitly known. These schemes have the property that their solutions do not have numerical instabilities.

It is important however to note that given a set of linearly independent functions,

$$\{y^i(t)\}; i = 1,2,..., N \quad (17)$$

It is always possible to construct an $N\,th$ order linear difference equation that has the corresponding discrete functions as solutions (Mickens, 1990). Let,

$$y_n^{(i)} \equiv y^{(i)}(t_n),\ t_n = (\Delta t)n = hn \quad (18)$$

Then the following determinant gives the required difference equation,

$$\begin{vmatrix} y_n & y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(k)} \\ y_{n+1} & y_{n+1}^{(1)} & y_{n+1}^{(2)} & \cdots & y_{n+1}^{(k)} \\ . & & & & \\ . & & & & \\ . & & & & \\ y_{n+k} & y_{n+k}^{(1)} & y_{n+k}^{(2)} & \cdots & y_{n+k}^{(k)} \end{vmatrix} = 0 \quad (19)$$

Consider the equation of the form (1), let us assume that the exact solution of the problem (1) at the point $t = t_n$ denoted by $y(t_n)$ has the same general solution with the numerical solution of the difference equation at the same point $t = t_n$ denoted by y_n. Thus, from equation (19), the corresponding difference equation is given by,

$$\begin{vmatrix} y_n & y_n^{(1)} & y_n^{(2)} & y_n^{(3)} \\ y_{n+1} & y_{n+1}^{(1)} & y_{n+1}^{(2)} & y_{n+1}^{(3)} \\ y_{n+2} & y_{n+2}^{(1)} & y_{n+2}^{(2)} & y_{n+2}^{(3)} \\ y_{n+3} & y_{n+3}^{(1)} & y_{n+3}^{(2)} & y_{n+3}^{(3)} \end{vmatrix} = \begin{vmatrix} y_n & y(t_n) & y'(t_n) & y''(t_n) \\ y_{n+1} & y(t_{n+1}) & y'(t_{n+1}) & y''(t_{n+1}) \\ y_{n+2} & y(t_{n+2}) & y'(t_{n+2}) & y''(t_{n+2}) \\ y_{n+3} & y(t_{n+3}) & y'(t_{n+3}) & y''(t_{n+3}) \end{vmatrix} = 0 \qquad (20)$$

Evaluating the determinant of (20), we obtain

$$\begin{aligned} & y_{n+3}\begin{pmatrix} y''(t_n)y'(t_{n+1})y(t_{n+2}) - y''(t_n)y(t_{n+1})y'(t_{n+2}) \\ - y'(t_n)y''(t_{n+1})y(t_{n+2}) + y'(t_n)y(t_{n+1})y''(t_{n+2}) \\ + y(t_n)y''(t_{n+1})y'(t_{n+2}) - y(t_n)y'(t_{n+1})y''(t_{n+2}) \end{pmatrix} + y_{n+2}\begin{pmatrix} y(t_n)y'(t_{n+1})y''(t_{n+3}) - y(t_n)y''(t_{n+1})y'(t_{n+3}) \\ - y'(t_n)y(t_{n+1})y''(t_{n+3}) + y'(t_n)y''(t_{n+1})y'(t_{n+3}) \\ + y''(t_n)y(t_{n+1})y'(t_{n+3}) - y''(t_n)y'(t_{n+1})y(t_{n+3}) \end{pmatrix} \\ & + y_{n+1}\begin{pmatrix} y(t_n)y''(t_{n+2})y'(t_{n+3}) - y(t_n)y'(t_{n+2})y''(t_{n+3}) \\ + y'(t_n)y(t_{n+2})y''(t_{n+3}) - y'(t_n)y''(t_{n+2})y(t_{n+3}) \\ y''(t_n)y(t_{n+2})y'(t_{n+3}) + y''(t_n)y'(t_{n+2})y(t_{n+3}) \end{pmatrix} + y_n\begin{pmatrix} y(t_{n+1})y'(t_{n+2})y''(t_{n+3}) - y(t_{n+1})y''(t_{n+2})y'(t_{n+3}) \\ - y'(t_{n+1})y(t_{n+2})y''(t_{n+3}) + y'(t_{n+1})y''(t_{n+2})y(t_{n+3}) \\ + y''(t_{n+1})y(t_{n+2})y'(t_{n+3}) - y''(t_{n+1})y'(t_{n+2})y(t_{n+3}) \end{pmatrix} = 0 \end{aligned} \qquad (21)$$

Solving (21) for y_{n+3}, we obtain

$$y_{n+3} = -\frac{\left[\begin{aligned} & y_{n+2}\begin{pmatrix} y(t_n)y'(t_{n+1})y''(t_{n+3}) - y(t_n)y''(t_{n+1})y'(t_{n+3}) \\ - y'(t_n)y(t_{n+1})y''(t_{n+3}) + y'(t_n)y''(t_{n+1})y'(t_{n+3}) \\ + y''(t_n)y(t_{n+1})y'(t_{n+3}) - y''(t_n)y'(t_{n+1})y(t_{n+3}) \end{pmatrix} \\ & + y_{n+1}\begin{pmatrix} y(t_n)y''(t_{n+2})y'(t_{n+3}) - y(t_n)y'(t_{n+2})y''(t_{n+3}) \\ + y'(t_n)y(t_{n+2})y''(t_{n+3}) - y'(t_n)y''(t_{n+2})y(t_{n+3}) \\ y''(t_n)y(t_{n+2})y'(t_{n+3}) + y''(t_n)y'(t_{n+2})y(t_{n+3}) \end{pmatrix} \\ & + y_n\begin{pmatrix} y(t_{n+1})y'(t_{n+2})y''(t_{n+3}) - y(t_{n+1})y''(t_{n+2})y'(t_{n+3}) \\ - y'(t_{n+1})y(t_{n+2})y''(t_{n+3}) + y'(t_{n+1})y''(t_{n+2})y(t_{n+3}) \\ + y''(t_{n+1})y(t_{n+2})y'(t_{n+3}) - y''(t_{n+1})y'(t_{n+2})y(t_{n+3}) \end{pmatrix} \end{aligned}\right]}{\begin{pmatrix} y''(t_n)y'(t_{n+1})y(t_{n+2}) - y''(t_n)y(t_{n+1})y'(t_{n+2}) \\ - y'(t_n)y''(t_{n+1})y(t_{n+2}) + y'(t_n)y(t_{n+1})y''(t_{n+2}) \\ + y(t_n)y''(t_{n+1})y'(t_{n+2}) - y(t_n)y'(t_{n+1})y''(t_{n+2}) \end{pmatrix}} \qquad (22)$$

Shifting downward the index n by two units, we get

$$y_{n+1}=-\frac{\left[\begin{array}{l} y_n\begin{pmatrix} y(t_{n-2})y'(t_{n-1})y''(t_{n+1})-y(t_{n-2})y''(t_{n-1})y'(t_{n+1}) \\ -y'(t_{n-2})y(t_{n-1})y''(t_{n+1})+y'(t_{n-2})y''(t_{n-1})y'(t_{n+1}) \\ +y''(t_{n-2})y(t_{n-1})y'(t_{n+1})-y''(t_{n-2})y'(t_{n-1})y(t_{n+1}) \end{pmatrix} \\ +y_{n-1}\begin{pmatrix} y(t_{n-2})y''(t_n)y'(t_{n+1})-y(t_{n-2})y'(t_n)y''(t_{n+1}) \\ +y'(t_{n-2})y(t_n)y''(t_{n+1})-y'(t_{n-2})y''(t_n)y(t_{n+1}) \\ y''(t_{n-2})y(t_n)y'(t_{n+1})+y''(t_{n-2})y'(t_n)y(t_{n+1}) \end{pmatrix} \\ +y_{n-2}\begin{pmatrix} y(t_{n-1})y'(t_n)y''(t_{n+1})-y(t_{n-1})y''(t_n)y'(t_{n+1}) \\ -y'(t_{n-1})y(t_n)y''(t_{n+1})+y'(t_{n-1})y''(t_n)y(t_{n+1}) \\ +y''(t_{n-1})y(t_n)y'(t_{n+1})-y''(t_{n-1})y'(t_n)y(t_{n+1}) \end{pmatrix} \end{array}\right]}{\begin{pmatrix} y''(t_{n-2})y'(t_{n-1})y(t_n)-y''(t_{n-2})y(t_{n-1})y'(t_n) \\ -y'(t_{n-2})y''(t_{n-1})y(t_n)+y'(t_{n-2})y(t_{n-1})y''(t_n) \\ +y(t_{n-2})y''(t_{n-1})y'(t_n)-y(t_{n-2})y'(t_{n-1})y''(t_n) \end{pmatrix}} \qquad (23)$$

Equation (23) is the EFDS capable of solving any problem of the form (1). It is important to note that the EFDS (23) is of the form (13)

Results

The EFDS developed in this research shall be adopted in solving some modeled real-life oscillatory problems of the form (1). The following notations shall be used in the tables below.

ESJ-Absolute error in Sunday (2018)

Eval $t/\sec$- Evaluation time per seconds for computation at each stage

Problem 1:

Consider the third order oscillatory problem,

$$\frac{d^3y}{dt^3}=-y'(t),\quad y(0)=0, y'(0)=1, y''(0)=2 \qquad (24)$$

whose exact solution is given by,

$$y(t)=2(1-\cos t)+\sin t \qquad (25)$$

Source: Sunday (2018)

On the application of the newly derived EFDS (23) on Problem 1 we obtain the result presented in Table 1 below.

Table 1: Showing the result for problem 1

t	Exact Solution	Computed Solution	Error	ESJ	Eval $t/\sec$
0.1000	0.109825086090777	0.109825086090777	0.000000e+000	3.7470e-016	0.1121
0.2000	0.238536175112578	0.238536175112578	0.000000e+000	8.3267e-016	0.1281
0.3000	0.384847228410128	0.384847228410128	0.000000e+000	1.3878e-015	0.1492
0.4000	0.547296354302881	0.547296354302881	0.000000e+000	1.4433e-015	0.1664
0.5000	0.724260414823458	0.724260414823458	0.000000e+000	1.5543e-015	1.1884
0.6000	0.913971243575679	0.913971243575679	0.000000e+000	1.9984e-015	1.2051
0.7000	1.114533312668715	1.114533312668715	0.000000e+000	2.8866e-015	1.2222
0.8000	1.323942672205193	1.323942672205193	0.000000e+000	4.4409e-015	1.2553
0.9000	1.540106973086156	1.540106973086156	0.000000e+000	3.5527e-015	1.2726
1.0000	1.760866373071619	1.760866373071619	0.000000e+000	5.3291e-015	1.2947

Problem 2:
Consider the third order oscillatory problem,

$$\frac{d^3y}{dt^3} = y''(t) - y'(t) + y(t), \quad y(0) = 1, y'(0) = 0, y''(0) = -1, \; h = 0.01 \tag{26}$$

whose exact solution is given by,

$$y(t) = \cos t \tag{27}$$

Source: Sunday (2018)

On the application of the newly derived EFDS (23) on Problem 2 we obtain the result presented in Table 2 below.

Table 2: Showing the result for problem 2

t	Exact Solution	Computed Solution	Error	ESJ	Eval $t/\sec$
0.0100	0.999950000416665	0.999950000416665	0.000000e+000	1.1102e-016	0.0190
0.0200	0.999800006666578	0.999800006666578	0.000000e+000	1.3323e-015	0.0194
0.0300	0.999550033748988	0.999550033748988	0.000000e+000	9.6589e-015	0.0197
0.0400	0.999200106660978	0.999200106660978	0.000000e+000	3.2974e-014	0.0201
0.0500	0.998750260394966	0.998750260394966	0.000000e+000	8.2379e-014	0.0207

Problem 3:
Consider the third order oscillatory problem,

$$\frac{d^3y}{dt^3} = 3\sin t, \quad y(0) = 1, y'(0) = 0, y''(0) = -2 \tag{28}$$

whose exact solution is given by,

$$y(t) = 3\cos t + \frac{t^2}{2} - 2 \tag{29}$$

Source: Sunday (2018)

On the application of the newly derived EFDS (23) on Problem 3 we obtain the result presented in Table 3 below.

Table 3: Showing the result for problem 3

t	Exact Solution	Computed Solution	Error	ESJ	Eval $t/\sec$
0.1000	0.990012495834077	0.990012495834077	0.000000e+000	3.3307e-016	0.0050
0.2000	0.960199733523725	0.960199733523725	0.000000e+000	3.3307e-016	0.0061
0.3000	0.911009467376818	0.911009467376818	0.000000e+000	3.3307e-016	0.0069
0.4000	0.843182982008655	0.843182982008655	0.000000e+000	1.1102e-016	0.0072
0.5000	0.757747685671118	0.757747685671118	0.000000e+000	1.1102e-016	0.0077
0.6000	0.656006844729034	0.656006844729034	0.000000e+000	4.4409e-016	0.0081
0.7000	0.539526561853465	0.539526561853465	0.000000e+000	5.5511e-016	0.0085
0.8000	0.410120128041496	0.410120128041496	0.000000e+000	5.5511e-016	0.0090
0.9000	0.269829904811993	0.269829904811993	0.000000e+000	7.2164e-016	0.0093
1.0000	0.120906917604418	0.120906917604418	0.000000e+000	1.0547e-015	0.0097

Problem 4:
Consider the third order oscillatory problem,

$$\frac{d^3y}{dt^3} = -4y'(t) + t, \quad y(0) = y'(0) = 0, y''(0) = 1 \qquad (30)$$

whose exact solution is given by,

$$y(t) = \left(\frac{3}{16}\right)(1 - \cos 2t) + \left(\frac{1}{8}\right)t^2 \qquad (31)$$

Source: Sunday (2018)

On the application of the newly derived EFDS (23) on Problem 4 we obtain the result presented in Table 4 below.

Table 4: Showing the result for problem 4

t	Exact Solution	Computed Solution	Error	ESJ	Eval $t/\sec$
0.1000	0.004987516654767	0.004987516654767	0.000000e+000	8.3209e-013	0.0193
0.2000	0.019801063624459	0.019801063624459	0.000000e+000	3.4752e-012	0.0241
0.3000	0.043999572204435	0.043999572204435	0.000000e+000	7.8178e-012	0.0387
0.4000	0.076867491997407	0.076867491997407	0.000000e+000	1.3681e-011	0.0533
0.5000	0.117443317649724	0.117443317649724	0.000000e+000	2.0825e-011	0.0678
0.6000	0.164557921035624	0.164557921035624	0.000000e+000	2.8962e-011	0.0786
0.7000	0.216881160706205	0.216881160706205	0.000000e+000	3.7764e-011	0.0864
0.8000	0.272974910431492	0.272974910431492	0.000000e+000	4.6879e-011	0.0901
0.9000	0.331350392754954	0.331350392754954	0.000000e+000	5.5941e-011	0.1001
1.0000	0.390527531852590	0.390527531852590	0.000000e+000	6.4592e-011	0.1009

Results and Discussion

The results obtained in Tables 1-4 clearly show that the EFDS in equation (23) is computationally reliable and efficient. This is because the computed solution matches exactly with the exact solution. In fact, the method performed better than the ones with which we compared our results. The method is also efficient because from the tables, the evaluation times per seconds are very small. This shows that the method generates results very fast (in microseconds).

Conclusion

A new EFDS has been developed in this paper for the solution of third order oscillatory problem. The method developed was applied on some modeled problems and from the results obtained, it is clear that the method is computationally reliable. The analysis of the method derived was also carried out. A major advantage of the method is that it does not exhibit any numerical instability.

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