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Special Computation of Duffing Oscillators via Non-Standard Finite Difference Method

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Abstract

In this paper, a new Non-Standard Finite Difference Method (NSFDM) is constructed for the computation of a special class of nonlinear differential equations called the Duffing oscillators. The nonstandard method is formulated by approximating the nonlinear terms nonlocally in the Duffing oscillator and also by reconstructing the denominator functions. The paper went further to analyze the perturbation solutions of Duffing oscillators using the nonstandard finite difference method. The need for this method came up due to some shortcomings of existing methods in which the qualitative properties of the exact solutions are not usually transferred to the numerical (approximate) solutions. The approach developed in this research has the property that its solution does not exhibit numerical instabilities in view of the results generated.

Keywords: Computation; denominator function; Duffing oscillator; nonlinear; NSFDM

Introduction

Duffing oscillator is one of the most significant and classical nonlinear ordinary differential equations in view of its diverse applications in science and engineering, Sunday (2017). Little wonder, it has received remarkable attention due to its variety of applications in science and engineering. The Duffing oscillators are applied in weak signal detection (Abolfazl and Hadi, 2011), magneto-elastic mechanical systems (Guckenheimer and Holmes, 1983), large amplitude oscillation of centrifugal governor systems (Younesian *et. al.*, 2011), nonlinear vibration of beams and plates (Bakhtiari-Nejad and Nazari, (2009), fluid flow induced vibration (Srinil and Zanganeh, 2012), among others. Given its characteristic of oscillation and chaotic nature, many scientists are inspired by this nonlinear differential equation since it replicates similar dynamics in our natural world.

In this paper, we shall consider a computational method for the simulation of Duffing oscillators of the form;

$$y''(t)+\eta y'(t)+\mu y(t)+\in y^{3}(t)=f(t) \quad (1)$$

with initial conditions,

$$y(0)=\alpha,\ y'(0)=\beta \quad (2)$$

where $\eta,\ \mu,\ \in,\ \alpha\ and\ \beta$ are real constants and $f(t)$ is a real-valued function. We shall assume that equation (1) satisfy the existence and uniqueness theorem stated below.

Theorem 1 (Wend, 1967)

Let,

$$u^{(n)}=f(x,u,u',...,u^{(n-1)}),\ u^{(k)}(x_0)=c_k \quad (3)$$

$k=0,1,...,(n-1)$, *u and f are scalars. Let $\Re$ be the region defined by the inequalities* $x_0\le x\le x_0+a,\ \left|s_j-c_j\right|\le b,\ j=0,1,...,(n-1)$, $(a>0,b>0)$. *Suppose the function* $f(x,s_0,s_1,...,s_{n-1})$ *is defined in $\Re$ and in addition:*

(i) f *is non-negative and non-decreasing in each of* $x,s_0,s_1,...,s_{n-1}$ *in* $\Re$

(ii) $f(x, c_0, c_1, ..., c_{n-1}) > 0$, *for* $x_0 \le x \le x_0 + a$, *and*

(iii) $c_k \ge 0,\ k = 0,1,...,n-1$

Then, the initial value problem (1) and (2) has a unique solution in $\Re$.

It is important to note that the Duffing equation is a simple model that shows different types of oscillations such as chaos and limit cycles. The terms associated with the system in equation (1) as given by [1] are;

$y'(t)$: small damping

η: ratio (coefficient) of viscous damping (it controls the size of damping)

$\mu y(t) + \in y^3(t)$: nonlinear restoring force acting like a hard spring (with μ controlling the size of stiffness and $\in$ controlling the size of nonlinearity)

$f(t)$: small periodic force

Duffing oscillators are routinely associated with damping in physical systems (Sunday, 2017), where damping is defined as an influence within or upon oscillatory system that has the effect of reducing, restricting or preventing its oscillation.

Several methods have been proposed in literature for the computation of problems of the form (1). These methods include; Hybrid method (Sunday, 2017), Laplace decomposition method (Yusufoglu, 2006), restarted Adomian decomposition method (Vahidi, Azimzadeh and Mohammadifar, 2012), differential transform method (Tabatabaei and Gunerhan, 2014), modified differential transform method (Nourazar and Mirzabeigy, 2013), improved Taylor matrix method (Berna and Hehmet, 2013), variational iteration method (He, 1999; 2000), modified variational iteration method (Goharee and Bobolian, 2014), Trigonometrically fitted Obrechkoff method (Shokri *et. al.*, 2015), among others.

In recent years, to get reliable results with less effort, researchers have applied NSFDMs to solved differential equations and they obtained competitive results than those of the existing methods. These authors include, Mickens (1990, 1994, 1999), Anguelov and Lubuma (2000, 2003), Ibijola and Sunday (2010), Sunday (2010), Sunday *et. al*. (2011, 2015), among others.

Thus, in this paper we shall be interested in formulating a new nonstandard finite difference method of the form

$$y_{n+1} = F(h, y_n) \qquad (4)$$

for the computation of Duffing oscillators of the form (1). Note that equation (4) is the general form of nonstandard finite difference methods.

The Nonstandard Finite Difference Modeling Rules

Definition 2.1 Anguelov and Lubuma (2001)

A finite difference scheme is called **non-standard finite difference method**, if at least one of the following conditions is met;

i) in the discrete derivative, the traditional denominator is replaced by a non-negative function ϕ such that,

$$\phi(h) = h + o(h^2),\ as\, h \to 0 \qquad (5)$$

ii) non-linear terms that occur in the differential equation are approximated in a non-local way i.e. by a suitable functions of several points of the mesh. For example, $y^2 \approx y_n y_{n+1}, y_{n-1} y_n,\ y^3 \approx y_{n-1} y_n y_{n+1}, y_n^2 y_{n+1}$

We shall employ the following collection of rules set by Mickens (1994) in developing NSFDM for Duffing oscillators.

- The order of the discrete derivatives must be exactly equal to the order of the corresponding derivatives of the differential equation.
- Denominator function for the discrete derivatives must be expressed in terms of more complicated function of the step-sizes than those conventionally used.
- The non-linear terms must in general be modeled (approximated) non-locally on the computational grid or lattice in many

different ways. The non-linear terms y^2, y^3 can be modeled as follows

$$y^2 \cong y_n y_{n+1} \quad (6)$$

$$y^2 \cong y_n (\frac{y_{n+1}+y_n}{2}) \quad (7)$$

$$y^3 \cong y_n^2 y_{n+1} \quad (8)$$

$$y^3 \cong y_n^2 (\frac{y_{n+1}+y_n}{2}) \quad (9)$$

$$y^3 \cong y_{n-1} y_n y_{n+1} \quad (10)$$

The particular form selected from equations (6) to (10) depends on the full discrete model.

- Special solutions of the differential equations should also be accompanied by special discrete solutions of the finite-difference models. For instance, an ordinary differential equation for which the substitution, $t \to -t$, leaves the equation invariant. If the discrete model does not also have this property, then numerical instabilities may occur.
- The finite-difference equation should not have solutions that do not correspond exactly to the solution of the differential equations.

For the purpose of this work, we shall assume that the function $F(h, y)$ in (4) has continuous derivatives with respect to both variables for $h>0$, $y \in R$ and that;

$$F(0, y) = y, \ \frac{\partial F(0, y)}{\partial h} = f(y) \quad (11)$$

It is necessary to note that consistency implies (11) if y is the solution of the differential equation (1).

For completeness, we give the standard discrete representation for first and second derivatives as

$$\frac{dy}{dt} \to \begin{cases} \frac{y_{n+1}-y_n}{h} \\ \frac{y_n - y_{n-1}}{h} \\ \frac{y_{n+1}-2y_{n-2}}{2h} \end{cases} \quad (12)$$

$$\frac{d^2y}{dt^2} \to \frac{y_{n+1}-2y_n+y_{n-1}}{h^2} \quad (13)$$

Theorem 2 Anguelov and Lubuma (2003)
The finite difference scheme (4) is stable with respect to monotone dependence on initial value, if

$$\frac{\partial F(h, y)}{\partial y} \geq 0, y \in R, h>0 \quad (14)$$

Mathematical Formulation of Non-Standard Finite Difference Method for Duffing Oscillators

We shall formulate a NSFDM for Duffing oscillators of the form (1). This is achieved by nonlocal representation of the nonlinear term y^3 in equation (1) as follows,

$$\frac{y_{n+1}-2y_n+y_{n-1}}{h^2} + \eta\left(\frac{y_{n+1}-y_n}{h}\right) + \mu y_n + \in y_{n-1} y_n y_{n+1} = f(t) \quad (15)$$

That is, the nonlinear term y^3 in equation (1) is approximated by $y^3 \approx y_{n-1} y_n y_{n+1}$. From equation (15),

$$y_{n+1} - 2y_n + y_{n-1} + \eta h(y_{n+1} - y_n) + \mu h^2 y_n + \in h^2 y_{n-1} y_n y_{n+1} = h^2 f(t)$$

$$y_{n+1}(1 + \eta h + \in h^2 y_{n-1} y_n) = y_n(2 + \eta h - \mu h^2) - y_{n-1} + h^2 f(t)$$

Thus,

$$y_{n+1} = \frac{y_n(2 + \eta h - \mu h^2) - y_{n-1} + h^2 f(t)}{1 + \eta h + \in h^2 y_{n-1} y_n} \quad (16)$$

It is important to note that y_{n+1} is the value of the solution at $(n+1)^{th}$ time step, y_n is the value of the solution at n^{th} time step and h is the time stepping parameter. Equation (16) is a NSFDM (with trivial denominator) capable of solving Duffing oscillators of the form (1).

A more efficient method can be developed by replacing the denominator h in (16) with a denominator function $\phi(h)$ so that $\phi(h) \to 0 \; as \; h \to 0$. This nontrivial denominator helps in maintaining the positivity and stability of the solution. For the problems of the form (1), we approximate the denominator h as,

$$\phi(h) = 1 - e^{-h} \qquad (17)$$

Substituting (17) in (16), we get

$$y_{n+1} = \frac{y_n\left[2+\eta\left(1-e^{-h}\right)-\mu\left(1-e^{-h}\right)^2\right]-y_{n-1}+\left(1-e^{-h}\right)^2 f(t)}{1+\eta\left(1-e^{-h}\right)^2+\in\left(1-e^{-h}\right)^2 y_{n-1}y_n} \qquad (18)$$

Equation (18) is the NSFDM (with nontrivial denominator) capable of solving Duffing oscillators of the form (1). The nontrivial denominator function introduced in (18) helps in overcoming the unstable behavior of the NSFDM (with trivial denominator) in (16).

Analysis of Perturbation Solutions of Duffing Oscillators using the Nonstandard Finite Difference Technique

Suppose the Duffing oscillator in equation (1) can be written in the form of nonlinear second-order difference equation

$$\Gamma y_n = \in f\left(y_{n+1}, y_n, y_{n-1}\right) \qquad (19)$$

where $\in$ is a parameter satisfying the condition

$$0 < \in << 1 \qquad (20)$$

and the operator Γ is defined by the relation

$$\Gamma y_n \equiv \frac{y_{n+1} - 2y_n + y_{n-1}}{4\sin^2\left(\frac{h}{2}\right)} + y_n \qquad (21)$$

A multi-discrete-variable procedure is constructed to obtain the perturbation solutions to equation (1), see Mickens (1994). Firstly, two discrete variables $n \; and \; s = \in n$ are introduced and the solution to (1) is assumed to have the form,

$$y_n \equiv y(n,s,\in) = y_0(n,s) + \in y_1(n,s) + O(\in^2) \qquad (22)$$

where y_n is assumed to have at least a first partial derivative with respect to s. On the basis of these assumptions, we have

$$y_{n+1} = y(n+1, s+\in, \in) = y_0(n+1, s+\in) + \in y_1(n+1, s+\in) + O(\in^2) \qquad (23)$$

$$y(n+1, s+\in) = y_0(n+1, s) + \in \frac{\partial y_0(n+1,s)}{\partial s} + O(\in^2) \qquad (24)$$

$$y_1(n+1, s+\in) = y_1(n+1, s) + O(\in) \qquad (25)$$

and

$$y_{n+1} = y_0(k+1, s) + \in\left[y_1(n+1,s) + \frac{\partial y_0(n+1,s)}{\partial s}\right] + O(\in^2) \qquad (26)$$

$$y_{n-1} = y_0(k-1, s) + \in\left[y_1(n-1,s) - \frac{\partial y_0(n-1,s)}{\partial s}\right] + O(\in^2) \qquad (27)$$

Substituting equations (22), (26) and (27) into equation (19), and setting the coefficients of the $\in^0 \; and \in^1$ terms equal to zero, gives the following determinant equations for the unknown functions $y_0(n,s) \; and \; y_1(n, s)$

$$\Gamma y_0(n,s) = 0 \qquad (28)$$

$$\Gamma y_1(n,s) = \frac{1}{4\sin^2\left(\frac{h}{2}\right)}\left[\frac{\partial y_0(n-1,s)}{\partial s} - \frac{\partial y_0(n+1,s)}{\partial s}\right] + f\left[y_0(n+1,s), y_0(n,s), y_0(n-1,s)\right] \quad (29)$$

The first equation has the general solution

$$y_0(n,s) = A(s)\cos(hk) + B(s)\sin(hk) \quad (30)$$

where $A(s)$ *and* $B(s)$ are unknown functions.

Results

Numerical Experiments

We shall apply the newly developed nonstandard finite difference method for the computation of Duffing oscillators of the form (1).

The following notation shall be used in the Tables below:

ESZG-Absolute error in Sunday, Zirra and Gandafa (2017)

Problem 1:

Consider the undamped Duffing oscillator,

$$y''(t) + y(t) + y^3(t) = \left(\cos t + \varepsilon \sin 10t\right)^3 - 99\varepsilon \sin 10t \quad (31)$$

with the initial conditions,

$$y(0) = 1,\ y'(0) = 10\varepsilon \quad (32)$$

where $\varepsilon = 10^{-10}$. The exact solution is given by,

$$y(t) = \cos t + \varepsilon \sin 10t \quad (33)$$

This equation describes a periodic motion of low frequency with a small perturbation of high frequency.

Source: Sunday, Zirra and Gandafa (2017)

On the application of the newly formulated NSFDM in equation (18) on Problem 1, we obtain the result presented in Table 1 below.

Table 1: Showing the results for problem 1 in comparison with the absolute errors in Sunday, Zirra and Gandafa (2017)

t	Exact Solution	Computed Solution	Error	ESZG	Time/s
0.0025	0.9999968750041274	0.9999968750041274	0.000000e+000	0.000000e+000	0.1039
0.0050	0.9999875000310395	0.9999875000310395	0.000000e+000	0.000000e+000	0.1348
0.0075	0.9999718751393287	0.9999718751393287	0.000000e+000	1.110223e-016	0.1736
0.0100	0.9999500004266486	0.9999500004266486	0.000000e+000	0.000000e+000	0.2112
0.0125	0.9999218760297148	0.9999218760297148	0.000000e+000	0.000000e+000	0.2121
0.0150	0.9998875021243030	0.9998875021243030	0.000000e+000	1.110223e-016	0.2127
0.0175	0.9998468789252486	0.9998468789252486	0.000000e+000	1.110223e-016	0.2133
0.0200	0.9998000066864446	0.9998000066864446	0.000000e+000	2.220446e-016	0.2140
0.0225	0.9997468857008414	0.9997468857008414	0.000000e+000	1.110223e-016	0.2146
0.0250	0.9996875163004431	0.9996875163004431	0.000000e+000	0.000000e+000	0.2152
0.0275	0.9996218988563066	0.9996218988563066	0.000000e+000	0.000000e+000	0.2160

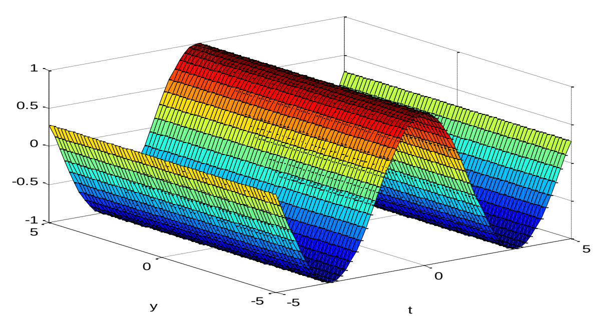

Figure1: Graphical result showing the oscillatory nature of Problem 1

Problem 2:

Consider the following undamped Duffing oscillator of the form;

$$y''(t)+y(t)+y^3(t)=B\cos\Omega t \quad (34)$$

with initial conditions,

$$y(0)=\alpha,\ y'(0)=0 \quad (35)$$

where,

$\alpha=0.200426728067, B=0.002, \Omega=1.01$

The exact solution to the problem is

$$y(t)=\sum_{i=0}^{3}A_{2i+1}Cos\left((2i+1)\Omega t\right) \quad (36)$$

where,

$$\begin{Bmatrix}A_1,A_3,A_5,\\A_7,A_9\end{Bmatrix}=\begin{Bmatrix}0.200179477536,0.0024946143,0.000000304014,\\0.000000000374,0.000000000000\end{Bmatrix}$$

Source: Sunday, Zirra and Gandafa (2017)

On the application of the newly formulated NSFDM in equation (18) on Problem 2, we obtain the result presented in Table 2 below.

Table 2: Comparison of the end-point absolute errors in Sunday, Zirra and Gandafa (2017) with that of the new nonstandard finite difference method

h	Error	EJS
$\frac{M}{500}$	0.000000e+000	4.846124e-015
$\frac{M}{1000}$	0.000000e+000	2.148108e-014
$\frac{M}{2000}$	0.000000e+000	9.221651e-014
$\frac{M}{3000}$	0.000000e+000	2.008060e-014
$\frac{M}{4000}$	0.000000e+000	2.930989e-014
$\frac{M}{5000}$	0.000000e+000	3.613776e-014

Note: $M=10$ in Table 2 above.

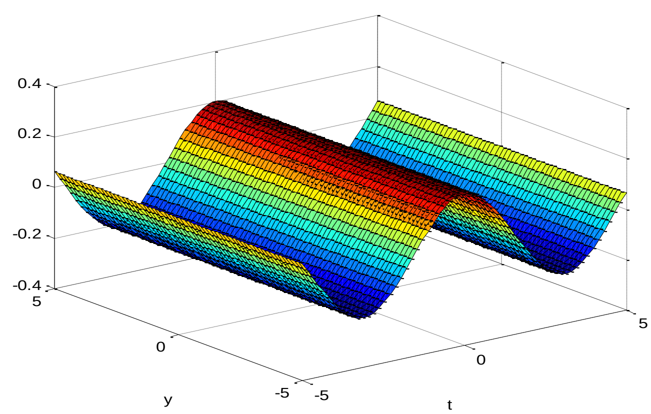

Figure 2: Graphical result showing the oscillatory nature of Problem 2

Problem 3:
Consider the damped Duffing oscillator,

$$y''(t)+y'(t)+y(t)+y^3(t)=\cos^3(t)-\sin(t) \quad (37)$$

whose initial conditions are,

$$y(0)=1,\ y'(0)=0 \quad (38)$$

The exact solution is given by,

$$y(t)=\cos(t) \quad (39)$$

Source: Sunday, Zirra and Gandafa (2017)

On the application of the newly formulated NSFDM in equation (18) on Problem 3, we obtain the result presented in Table 3 below.

Table 3: Showing the results for problem 3 in comparison with the absolute errors in Sunday, Zirra and Gandafa (2017)

t	Exact Solution	Computed Solution	Error	ESZG	Time/s
0.1000	0.9950041652780258	0.9950041652780258	0.000000e+000	1.110223e-016	0.0093
0.2000	0.9800665778412416	0.9800665778412416	0.000000e+000	2.220446e-016	0.0160
0.3000	0.9553364891256060	0.9553364891256060	0.000000e+000	0.000000e+000	0.0234
0.4000	0.9210609940028850	0.9210609940028850	0.000000e+000	2.220446e-016	0.0301
0.5000	0.8775825618903727	0.8775825618903727	0.000000e+000	1.110223e-016	0.0367
0.6000	0.8253356149096781	0.8253356149096781	0.000000e+000	1.110223e-016	0.0434
0.7000	0.7648421872844882	0.7648421872844882	0.000000e+000	1.110223e-016	0.0500
0.8000	0.6967067093471651	0.6967067093471651	0.000000e+000	1.110223e-016	0.0567
0.9000	0.6216099682706640	0.6216099682706640	0.000000e+000	1.110223e-016	0.0634
1.0000	0.5403023058681392	0.5403023058681392	0.000000e+000	2.220446e-016	0.0704

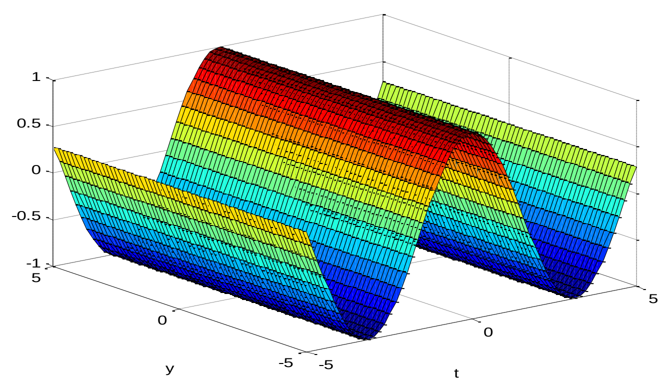

Figure 3: Graphical result showing the oscillatory nature of Problem 3

Problem 4:
Consider the undamped Duffing oscillator,

$$y''(t)+3y(t)+2y^3(t)=\cos(t)\sin(2t) \quad (40)$$

with the initial conditions,

$$y(0)=0,\ y'(0)=1 \quad (41)$$

The exact solution is given by,

$$y(t)=\sin(t) \quad (42)$$

Source: Sunday, Zirra and Gandafa (2017)

On the application of the newly formulated NSFDM in equation (18) on Problem 4, we obtain the result presented in Table 4 below.

Table 4: Showing the results for problem 4 in comparison with the absolute errors in Sunday, Zirra and Gandafa (2017)

t	Exact Solution	Computed Solution	Error	ESZG	Time/s
0.1000	0.0998334166468281	0.0998334166468281	0.000000e+000	1.387779e-017	0.0437
0.2000	0.1986693307950612	0.1986693307950612	0.000000e+000	0.000000e+000	0.0492
0.3000	0.2955202066613397	0.2955202066613397	0.000000e+000	1.110223e-016	0.0547
0.4000	0.3894183423086507	0.3894183423086507	0.000000e+000	2.220446e-016	0.0603
0.5000	0.4794255386042032	0.4794255386042032	0.000000e+000	2.775558e-016	0.0662
0.6000	0.5646424733950356	0.5646424733950356	0.000000e+000	3.330669e-016	0.0719
0.7000	0.6442176872376914	0.6442176872376914	0.000000e+000	5.551115e-016	0.0775
0.8000	0.7173560908995231	0.7173560908995231	0.000000e+000	5.551115e-016	0.0831
0.9000	0.7833269096274838	0.7833269096274838	0.000000e+000	8.881784e-016	0.0888
1.0000	0.8414709848078968	0.8414709848078968	0.000000e+000	6.661338e-016	0.0946

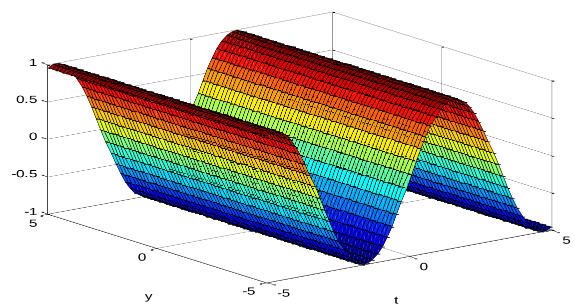

Figure 4: Graphical result showing the oscillatory nature of Problem 4

Discussion

We computed some Duffing oscillators with the aid of the newly derived nonstandard finite difference method and from the results obtained, it is obvious that the nonstandard finite difference method formulated is more efficient than the existing ones with which we compared our results.

Conclusion

A highly efficient nonstandard finite difference method has been formulated by nonlocal approximation of the nonlinear terms and the reformulation of the denominator function. The method developed was used to approximate Duffing oscillators and it is obvious from the results (numerical and graphical) obtained that the method is computationally reliable. The results show that the approximate solutions (obtained using the NSFDM) converge exactly to the exact solutions. The evaluation time per seconds obtained were also observed to be very small, showing that the method derived generates results very fast. The perturbation solution of the Duffing oscillator was also analyzed.

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