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# On the Derivation and Implementation of a One-Step Algorithm for Third Order Oscillatory Problems

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**Abstract**

In this paper, a one-step algorithm is derived and implemented for third order oscillatory problems. The derivation is carried out using the procedure of collocation and interpolation of power series basis function within a one-step integration interval $[x_n, x_{n+1}]$.The paper also analyzed some basic properties of the algorithm derived. The results obtained on the application of the one-step algorithm on some sampled modeled third order oscillatory problems show that the algorithm is computationally reliable and it performed better than the ones with which we compared our results with.

**Keywords:** Algorithm; Computation; Bounded solutions; One-step; Oscillation; Third-order
**2010 AMS Subject Classification**: 65L05; 65L06; 65D30

## Introduction

According to Sunday (2018), one of the most challenging equations being encountered nowadays is the oscillatory differential equations. This is because their solutions are composed of smooth varying and ‘nearly periodic’ functions, i.e. they are oscillations whose wave form and period varies slowly with time (relative to the period), and where the solution is sought over a very large number of cycles, Stetter (1994). For such problems, one cannot and does not want to follow the trajectories; instead one resort to finding their approximate solutions or the computation of their quasi-envelops.

Oscillatory problems have some of their Eigen values near the imaginary axis, and their solutions are oscillation processes with slowly varying amplitudes. The difficulty of solving such problems is explained by the necessity to ensure correct values of the amplitude and phase angle over many periods.

In this research, we shall derive and implement a one-step algorithm on third order oscillatory problems of the form,

$$y'''(t) = f(t.y, y', y''),\ y(t_0) = y_0, y'(t_0) = y_0',\ y''(t_0) = y_0'',\ t \in [t_0, t_n] \qquad (1)$$

where $t_0$ is the initial value/point, $y_0$ is the solution at $t_0$, $f$ is continuous within the interval of integration. It is assumed that (1) satisfies the existence and uniqueness theorem of differential equations. It is also assumed that the solutions to equations of the form (1) are bounded. It is important to state that a solution $y(t)$ to (1) is said to be bounded if,

$$\sup_{t \in \Re} \|y(t)\| < \infty \qquad (2)$$

It is important to state that (1) has a wide range of applications in engineering, thermodynamics and other real life problems. They are also applied in studying thin-film flows Duffy and Wilson (1997), chaotic systems Genesio and Tesi (1992), electromagnetic waves Lee, Fudziah and Norazak (2014), among other phenomenon.

A solution of (1) will be called oscillatory if it has infinity of zeros in $(0, \infty)$ and non-oscillatory if it has but a finite number of zeros in this interval, Hanan (1961). An equation is termed oscillatory if there exists at least one oscillatory solution and non-oscillatory if all its solutions are non-oscillatory. This latter definition is necessary since an (1) may be both oscillatory and non-oscillatory.

Some methods have been derived by authors to directly solve third order differential equations of the form (1), see the works of Adesanya, Udoh and Alkali (2012), Majid *et al.* (2012), Adesanya, Udoh and Ajileye (2013), Lee, Fudziah and Norazak (2014), Awoyemi, Kayode and Adoghe (2014), Mohammed and Adeniyi (2014), Yakusak, Akinyemi and Usman (2016), etc. Direct method for solving (1) has been reported to be more efficient than the method of reduction to system of first order differential equations, see Lee, Fudziah and Norazak (2014), Adesanya, Alkali and Sunday (2014), among others.

**Definition 1**

A differential equation is said to be oscillatory if,

(i) all the nontrivial solution of (1) have an infinite number of zeros (roots) on $x_0 \le x < \infty$, see Kanat (2006) and

(ii) it has at least one oscillating solution, Borowski and Borwein (2005)

**Definition 2** (Lambert, 1991)

A computational method is said to be A-stable if the whole of the left-half plane $\{z : \mathrm{Re}(z) \le 0\}$ is contained in the region $\{z : |\mathrm{Re}(z)| \le 1\}$, where $R(z)$ is called the stability polynomial of the method.

**Derivation of the One-step Algorithm**

A one-step algorithm shall be derived for the computation of third order oscillatory problems of the form (1). Thus, we shall employ power series approximate solution of the form,

$$y(t) = \sum_{j=0}^{r+s-1} a_j t^j \tag{3}$$

for the derivation of the one-step algorithm given by,

$$\mathbf{A}^{(0)}\mathbf{Y}_m^{(i)} = \sum_{i=0}^{1} \frac{(jh)^{(i)}}{i!} e_i y_n^{(i)} + h^{(3-i)}\left[\mathbf{d}_i f(y_n) + \mathbf{b}_i \mathbf{F}(\mathbf{Y}_m)\right] \tag{4}$$

where $r \text{ and } s$ (3) are the numbers of collocation and interpolation points respectively.

Equation (3) is differentiated three times and substituted into (1), that is,

$$f(t, y, y', y'') = \sum_{j=2}^{r+s-1} j(j-1)(j-2) a_j t^{j-3} \tag{5}$$

A grid of one-steplength is considered in this paper with a constant step size $h$ given by $h = t_{n+i} - t_n,\ i = 0,1$ and off-step points at $t_{n+\frac{1}{5}},\ t_{n+\frac{2}{5}}$ *and* $t_{n+\frac{3}{5}}$.

Interpolating (3) at point $t_{n+s},\ s = \frac{1}{5}\left(\frac{2}{5}\right)\frac{3}{5}$ and collocating (5) at points $t_{n+r},\ r = 0\left(\frac{1}{5}\right)1$, give a system of nonlinear equation of the form,

$$TA = U \tag{6}$$

where

$$A = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8]^T$$

$$U = \left[ y_{n+\frac{1}{5}} \quad y_{n+\frac{2}{5}} \quad y_{n+\frac{3}{5}} \quad f_n \quad f_{n+\frac{1}{5}} \quad f_{n+\frac{2}{5}} \quad f_{n+\frac{3}{5}} \quad f_{n+\frac{4}{5}} \quad f_{n+1} \right]^T$$

$$T=\begin{bmatrix} 1 & t_{n+\frac{1}{5}} & t^2_{n+\frac{1}{5}} & t^3_{n+\frac{1}{5}} & t^4_{n+\frac{1}{5}} & t^5_{n+\frac{1}{5}} & t^6_{n+\frac{1}{5}} & t^7_{n+\frac{1}{5}} & t^8_{n+\frac{1}{5}} \\ 1 & t_{n+\frac{2}{5}} & t^2_{n+\frac{2}{5}} & t^3_{n+\frac{2}{5}} & t^4_{n+\frac{2}{5}} & t^5_{n+\frac{2}{5}} & t^6_{n+\frac{2}{5}} & t^7_{n+\frac{2}{5}} & t^8_{n+\frac{2}{5}} \\ 1 & t_{n+\frac{3}{5}} & t^2_{n+\frac{3}{5}} & t^3_{n+\frac{3}{5}} & t^4_{n+\frac{3}{5}} & t^5_{n+\frac{3}{5}} & t^6_{n+\frac{3}{5}} & t^7_{n+\frac{3}{5}} & t^8_{n+\frac{3}{5}} \\ 0 & 0 & 0 & 6 & 24t_n & 60t_n^2 & 120t_n^3 & 210t_n^4 & 336t_n^5 \\ 0 & 0 & 0 & 6 & 24t_n & 60t^2_{n+\frac{1}{5}} & 120t^3_{n+\frac{1}{5}} & 210t^4_{n+\frac{1}{5}} & 336t^5_{n+\frac{1}{5}} \\ 0 & 0 & 0 & 6 & 24t_n & 60t^2_{n+\frac{2}{5}} & 120t^3_{n+\frac{2}{5}} & 210t^4_{n+\frac{2}{5}} & 336t^5_{n+\frac{2}{5}} \\ 0 & 0 & 0 & 6 & 24t_n & 60t^2_{n+\frac{3}{5}} & 120t^3_{n+\frac{3}{5}} & 210t^4_{n+\frac{3}{5}} & 336t^5_{n+\frac{3}{5}} \\ 0 & 0 & 0 & 6 & 24t_n & 60t^2_{n+\frac{4}{5}} & 120t^3_{n+\frac{4}{5}} & 210t^4_{n+\frac{4}{5}} & 336t^5_{n+\frac{4}{5}} \\ 0 & 0 & 0 & 6 & 24t_n & 60t^2_{n+1} & 120t^3_{n+1} & 210t^4_{n+1} & 336t^5_{n+1} \end{bmatrix}$$

Solving (6) for $a_j,\ j=0(1)8$ which are constants to be determined and putting back into (3) gives a one-step continuous algorithm of the form,

$$y(t)=\alpha_{\frac{1}{5}}(t)y_{n+\frac{1}{5}}+\alpha_{\frac{2}{5}}(t)y_{n+\frac{2}{5}}+\alpha_{\frac{3}{5}}(t)y_{n+\frac{3}{5}}+h^3\left[\sum_{j=0}^{1}\beta_j(t)f_{n+j}+\beta_s(t)f_{n+s}\right],\ s=\frac{1}{5}\left(\frac{1}{5}\right)\frac{3}{5} \qquad (7)$$

where $\alpha_s(t),\ \beta_j(t)\ and\ \beta_s(t)$ are expressed as functions of $x$ with

$$x=\frac{t-t_n}{h} \qquad (8)$$

to obtain the continuous form as follows,

$$\left.\begin{aligned}
&\alpha_{\frac{1}{5}}(t)=\frac{1}{2}(25x^2-25x+6)\\
&\alpha_{\frac{2}{5}}(t)=-25x^2+20x-3\\
&\alpha_{\frac{3}{5}}(t)=\frac{1}{2}(25x^2-15x+2)\\
&\beta_0(t)=-\frac{1}{5040000}\begin{pmatrix}390625x^8-1875000x^7+3718750x^6-39375000x^5\\+2397500x^4-840000x^3+158525x^2-13320x+252\end{pmatrix}\\
&\beta_{\frac{1}{5}}(t)=\frac{1}{5040000}\begin{pmatrix}1953125x^8-8750000x^7+15531250x^6-13475000x^5\\+5250000x^4-701175x^2+210700x-19068\end{pmatrix}\\
&\beta_{\frac{2}{5}}(t)=-\frac{1}{2520000}\begin{pmatrix}1953125x^8-8125000x^7+12906250x^6-9362500x^5\\+2625000x^4+33825x^2-71440x+11004\end{pmatrix}\\
&\beta_{\frac{3}{5}}(t)=\frac{1}{2520000}\begin{pmatrix}1953125x^8-7500000x^7+10718750x^6-6825000x^5\\+1750000x^4-57875x^2+1740x+756\end{pmatrix}\\
&\beta_{\frac{4}{5}}(t)=-\frac{1}{5040000}\begin{pmatrix}1953125x^8-6875000x^7+8968750x^6-5337500x^5\\+1312500x^4-41775x^2+1000x+588\end{pmatrix}\\
&\beta_1(t)=\frac{1}{5040000}\begin{pmatrix}390625x^8-1250000x^7+1531250x^6-875000x^5\\+210000x^4-6675x^2+220x+84\end{pmatrix}
\end{aligned}\right\} \quad (9)$$

Solving (7) for the independent solution gives a continuous algorithm of the form,

$$y(t)=\sum_{i=0}^{1}\frac{(jh)^i}{i!}y_n^{(i)}+h^3\left[\sum_{j=0}^{1}\sigma_j(t)f_{n+j}+\sigma_s(t)f_{n+s}\right],\ s=\frac{1}{5}\left(\frac{1}{5}\right)\frac{3}{5} \quad (10)$$

where

$$\left.\begin{aligned}
&\sigma_0(t)=-\frac{1}{8064}(625x^2-3000x^7+5950x^6-6300x^5+3836x^4-1344x^3)\\
&\sigma_{\frac{1}{5}}(t)=\frac{5}{8064}(625x^8-2800x^7+4970x^6-4312x^5+1680x^4)\\
&\sigma_{\frac{2}{5}}(t)=-\frac{5}{4032}(625x^8-2600x^7+4130x^6-2996x^5+840x^4)\\
&\sigma_{\frac{3}{5}}(t)=\frac{5}{4032}(625x^8-2400x^7+3430x^6-2184x^5+560x^4)\\
&\sigma_{\frac{4}{5}}(t)=-\frac{5}{8064}(625x^8-2200x^7+2870x^6-1708x^5+420x^4)\\
&\sigma_1(t)=\frac{1}{8064}(625x^8-2000x^7+2450x^6-1400x^5+336x^4)
\end{aligned}\right\} \quad (11)$$

and $t$ is as defined in equation (8).

Evaluating (10) at $t=\frac{1}{5}\left(\frac{1}{5}\right)1$, gives a discrete one-step algorithm of the form (1) where,

$$\mathbf{Y}_m^{(i)}=\begin{bmatrix}y_{n+\frac{1}{5}}^{(i)} & y_{n+\frac{2}{5}}^{(i)} & y_{n+\frac{3}{5}}^{(i)} & y_{n+\frac{4}{5}}^{(i)} & y_{n+1}^{(i)}\end{bmatrix}^T \qquad \mathbf{F}(\mathbf{Y}_m)=\begin{bmatrix}f_{n+\frac{1}{5}} & f_{n+\frac{2}{5}} & f_{n+\frac{3}{5}} & f_{n+\frac{4}{5}} & f_{n+1}\end{bmatrix}^T$$

$$y_n^{(i)} = \begin{bmatrix} y_{n-\frac{1}{5}}^{(i)} & y_{n-\frac{2}{5}}^{(i)} & y_{n-\frac{3}{4}}^{(i)} & y_{n-\frac{4}{5}}^{(i)} & y_n^{(i)} \end{bmatrix}^T$$

and $\mathbf{A}^{(0)}$ is a $5\times5$ identity matrix.

When $i=0$

$$e_0 = \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \end{bmatrix} \quad e_1 = \begin{bmatrix} 0&0&0&0&\frac{1}{5} \\ 0&0&0&0&\frac{2}{5} \\ 0&0&0&0&\frac{3}{5} \\ 0&0&0&0&\frac{4}{5} \\ 0&0&0&0&1 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0&0&0&0&\frac{1}{50} \\ 0&0&0&0&\frac{2}{25} \\ 0&0&0&0&\frac{9}{50} \\ 0&0&0&0&\frac{8}{25} \\ 0&0&0&0&\frac{1}{2} \end{bmatrix}$$

$$d_0 = \begin{bmatrix} 0&0&0&0&\frac{3929}{5040000} \\ 0&0&0&0&\frac{317}{78750} \\ 0&0&0&0&\frac{783}{80000} \\ 0&0&0&0&\frac{712}{39375} \\ 0&0&0&0&\frac{233}{8064} \end{bmatrix} \quad b_0 = \begin{bmatrix} \frac{199}{201600} & \frac{-1931}{2520000} & \frac{173}{360000} & \frac{-883}{5040000} & \frac{139}{5040000} \\ \frac{367}{39375} & \frac{-38}{7875} & \frac{122}{39375} & \frac{-89}{78750} & \frac{1}{5625} \\ \frac{16119}{560000} & \frac{-2187}{280000} & \frac{423}{56000} & \frac{-1539}{560000} & \frac{243}{560000} \\ \frac{2336}{39375} & \frac{-32}{5625} & \frac{704}{39375} & \frac{-8}{1575} & \frac{32}{39375} \\ \frac{815}{8064} & \frac{5}{4032} & \frac{155}{4032} & \frac{-5}{1152} & \frac{11}{8064} \end{bmatrix}$$

When $i=1$

$$e_1 = \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0&0&0&0&\frac{1}{5} \\ 0&0&0&0&\frac{2}{5} \\ 0&0&0&0&\frac{3}{5} \\ 0&0&0&0&\frac{4}{5} \\ 0&0&0&0&1 \end{bmatrix} \quad d_1 = \begin{bmatrix} 0&0&0&0&\frac{1231}{126000} \\ 0&0&0&0&\frac{71}{3150} \\ 0&0&0&0&\frac{123}{3500} \\ 0&0&0&0&\frac{376}{7875} \\ 0&0&0&0&\frac{61}{1008} \end{bmatrix}$$

$$b_1 = \begin{bmatrix} \frac{863}{50400} & \frac{-761}{63000} & \frac{941}{126000} & \frac{-341}{126000} & \frac{107}{252000} \\ \frac{544}{7875} & \frac{-37}{1575} & \frac{136}{7875} & \frac{-101}{15750} & \frac{8}{7875} \\ \frac{3501}{28000} & \frac{-9}{3500} & \frac{87}{2800} & \frac{-9}{875} & \frac{9}{5600} \\ \frac{1424}{7875} & \frac{176}{7875} & \frac{608}{7875} & \frac{-16}{1575} & \frac{16}{7875} \\ \frac{475}{2016} & \frac{25}{504} & \frac{125}{1008} & \frac{25}{1008} & \frac{11}{2016} \end{bmatrix}$$

When $i=2$

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad d_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{19}{288} \\ 0 & 0 & 0 & 0 & \frac{14}{225} \\ 0 & 0 & 0 & 0 & \frac{51}{800} \\ 0 & 0 & 0 & 0 & \frac{14}{225} \\ 0 & 0 & 0 & 0 & \frac{19}{288} \end{bmatrix}$$

$$b_2 = \begin{bmatrix} \frac{1427}{72000} & \frac{-133}{1200} & \frac{241}{3600} & \frac{-173}{7200} & \frac{3}{800} \\ \frac{43}{150} & \frac{7}{225} & \frac{7}{225} & \frac{-1}{75} & \frac{1}{450} \\ \frac{219}{800} & \frac{57}{400} & \frac{57}{400} & \frac{-21}{800} & \frac{3}{800} \\ \frac{64}{225} & \frac{8}{75} & \frac{64}{225} & \frac{14}{225} & 0 \\ \frac{25}{96} & \frac{25}{144} & \frac{25}{144} & \frac{25}{96} & \frac{19}{288} \end{bmatrix}$$

**Analysis of Basic Properties of the One-step Algorithm**

***Order of the One-step Algorithm***

Let the linear operator $\ell\{y(t):h\}$ be defined on (4) when $i=0$ such that

$$\ell\{y(t):h\} = \mathbf{A}^{(0)}\mathbf{Y}_m^{(i)} - \sum_{i=0}^{1} \frac{(jh)^{(i)}}{i!} e_i y_n^{(i)} + h^{(3-i)}\left[\mathbf{d}_i f(y_n) + \mathbf{b}_i \mathbf{F}(\mathbf{Y}_m)\right] \tag{12}$$

From (12), expanding $\mathbf{Y}_m$ *and* $\mathbf{F}(\mathbf{Y}_m)$ in Taylor's series and comparing the coefficients of $h$ gives

$$\ell\{y(t):h\} = C_0 y(t) + C_1 y'(t) + \ldots + C_p h^p y^p(t) + C_{p+1} h^{p+1} y^{p+1}(t) + C_{p+2} h^{p+2} y^{p+2}(t) + \ldots \tag{13}$$

**Definition 3** (Lambert, 1991)

The linear operator $\ell$ and the associated one-step algorithm (4) are said to be of order $p$ if $C_0 = C_1 = ... = C_p = C_{p+1} = C_{p+2} = 0,\ C_{p+3} \neq 0.$ $C_{p+3}$ is called the error constant and implies that the truncation error is given by $T_{n+k} = C_{p+3} h^{p+3} y^{p+3}(t) + O(h^{p+4})$

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{5}h\right)^j}{j!} - y_n - \frac{1}{5}hy_n' - \frac{1}{50}h^2 y_n'' - \frac{3929}{5040000}h^3 y_n''' - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \begin{bmatrix} \frac{199}{201600}\left(\frac{1}{5}\right)^j - \frac{1931}{2520000}\left(\frac{2}{5}\right)^j + \frac{173}{360000}\left(\frac{3}{5}\right)^j \\ -\frac{883}{5040000}\left(\frac{4}{5}\right)^j + \frac{139}{5040000}(1)^j \end{bmatrix} \\ \sum_{j=}^{\infty} \frac{\left(\frac{2}{5}h\right)^j}{j!} - y_n - \frac{2}{5}hy_n' - \frac{2}{25}h^2 y_n'' - \frac{317}{78750}h^3 y_n''' - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \begin{bmatrix} \frac{367}{39375}\left(\frac{1}{5}\right)^j - \frac{38}{7875}\left(\frac{2}{5}\right)^j + \frac{122}{39375}\left(\frac{3}{5}\right)^j \\ -\frac{89}{78750}\left(\frac{4}{5}\right)^j + \frac{1}{5625}(1)^j \end{bmatrix} \\ \sum_{j=0}^{\infty} \frac{\left(\frac{3}{5}h\right)^j}{j!} - y_n - \frac{3}{5}hy_n' - \frac{9}{50}h^2 y_n'' - \frac{783}{80000}h^3 y_n''' - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \begin{bmatrix} \frac{16119}{560000}\left(\frac{1}{5}\right)^j - \frac{2187}{280000}\left(\frac{2}{5}\right)^j + \frac{423}{56000}\left(\frac{3}{5}\right)^j \\ -\frac{1539}{560000}\left(\frac{4}{5}\right)^j + \frac{243}{560000}(1)^j \end{bmatrix} \\ \sum_{j=0}^{\infty} \frac{\left(\frac{4}{5}h\right)^j}{j!} - y_n - \frac{4}{5}hy_n' - \frac{8}{25}h^2 y_n'' - \frac{712}{39375}h^3 y_n''' - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \begin{bmatrix} \frac{2336}{39375}\left(\frac{1}{5}\right)^j - \frac{32}{5625}\left(\frac{2}{5}\right)^j + \frac{704}{39375}\left(\frac{3}{5}\right)^j \\ -\frac{8}{1575}\left(\frac{4}{5}\right)^j + \frac{32}{39375}(1)^j \end{bmatrix} \\ \sum_{j=0}^{\infty} \frac{(h)^j}{j!} - y_n - hy_n' - \frac{1}{2}h^2 y_n'' - \frac{233}{8064}h^3 y_n''' - \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \begin{bmatrix} \frac{815}{8064}\left(\frac{1}{5}\right)^j + \frac{5}{4032}\left(\frac{2}{5}\right)^j + \frac{155}{4032}\left(\frac{3}{5}\right)^j \\ -\frac{5}{1152}\left(\frac{4}{5}\right)^j + \frac{11}{8064}(1)^j \end{bmatrix} \end{bmatrix} = 0$$

Comparing the coefficients of $h$, the order $p$ of the one-step algorithm is given by $p = [6\ \ 6\ \ 6\ \ 6\ \ 6]^T$ and its error constant is given by $\left[-1.3840\times 10^{-9}\ \ -8.8674\times 10^{-9}\ \ -2.1909\times 10^{-8}\ \ -4.1032\times 10^{-8}\ \ -6.5697\times 10^{-8}\right]^T$

***Consistency of the One-step Algorithm***

A computational method is said to be consistent if its order $p \geq 1$. The one-step algorithm derived is consistent since it is of uniform order 6. Consistency controls the magnitude of the local truncation error committed at each stage of the computation, (Fatunla, 1988) .

***Zero-stability of the One-step Algorithm***

**Definition 4** (Fatunla, 1988)

A computational method is said to be zero-stable, if the roots $z_{s,} s = 1,2,...,k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z) = \det(zA^{(0)} - E)$ satisfies $|z_s| \le 1$ and every root satisfying $|z_s| = 1$ have multiplicity not exceeding the order of the differential equation. The first characteristic polynomial is given by,

$$\rho(z) = \left| z\begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1 \end{bmatrix} \right| = \begin{vmatrix} z&0&0&0&-1\\ 0&z&0&0&-1\\ 0&0&z&0&-1\\ 0&0&0&z&-1\\ 0&0&0&0&z-1 \end{vmatrix} = z^4(z-1)$$

Thus, solving for $z$ in

$$z^4(z-1) = 0 \qquad (14)$$

gives $z = 0,\ 0,\ 0,\ 0,\ 1$. Hence, the one-step algorithm (4) is said to be zero-stable.

***Convergence of the One-step Algorithm***

**Theorem 1** (Fatunla, 1988)

The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero-stable.

Thus, the computational method formulated is convergent.

***Region of Absolute Stability of the One-step Algorithm***

**Definition 5** (Yan, 2011)

Region of absolute stability is a region in the complex $z$ plane, where $z = \lambda h$ for which the method is absolutely stable. It is defined as those values of $z$ such that the numerical solutions of $y''' = -\lambda y$ satisfy $y_j \to 0\ as\ j \to \infty$ for any initial condition.

The stability polynomial of the newly derived one-step algorithm is given by,

$$\bar{h}(w) = -h^{15}\left(\frac{1}{215332031250000}w^5 + \frac{1697}{5167968750000000}w^4\right) - h^{12}\left(\frac{144761}{7751953125000000}w^5 + \frac{2083381}{918750000000000}w^4\right)$$
$$-h^9\left(\frac{1}{84000000}w^5 + \frac{235831}{75600000000}w^4\right) - h^6\left(\frac{1}{26250000}w^5 + \frac{203113}{126000000}w^4\right) - \frac{13}{60}h^3w^4 + w^5 - \frac{5}{2}w^4 \qquad (15)$$

On using the stability polynomial in (15), we obtain the stability region in the figure below.

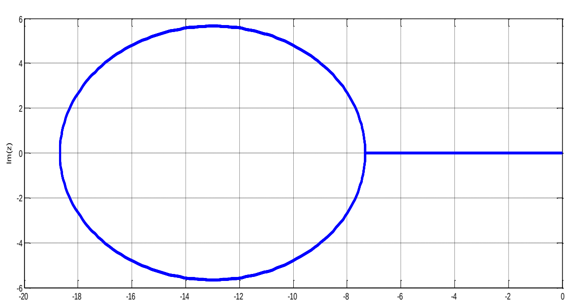


**Figure 1:** Region of absolute stability of the one-step algorithm

The region of absolute stability in Figure 1 is A-stable, since it contains the whole of the left-half complex plane of the figure. Note that the unstable region is the exterior of the curve (when the curve is on the negative plane) while the stability region is the interior of the curve.

***Implementation of the One-step Algorithm***

The one-step algorithm derived in this research can be used to implement higher differential equations of the form (1) without the need to reduce it to an equivalent system of first order. For the one-step algorithm derived which is of uniform order $p=6$, we use Taylor series expansion to calculate $y_{n+1}$ and its first, second and third derivatives up to order $p=6$.

$$y_{n+j} \equiv y(t_n + jh) \cong y(t_n) + jhy'(t_n) + \frac{(jh)^2}{2!} y''(t_n) + \frac{(jh)^3}{3!} f_n + \frac{(jh)^4}{4!} f_n' + \frac{(jh)^5}{5!} f_n'' + \frac{(jh)^6}{6!} f_n'''$$

$$y_{n+j}' \equiv y'(t_n + jh) \cong y'(t_n) + jhy''(t_n) + \frac{(jh)^2}{2!} f_n + \frac{(jh)^3}{3!} f_n' + \frac{(jh)^4}{4!} f_n'' + \frac{(jh)^5}{5!} f_n''' + \frac{(jh)^6}{6!} f_n^{iv}$$

$$y_{n+j}'' \equiv y''(t_n + jh) \cong y''(t_n) + jhf_n + \frac{(jh)^2}{2!} f_n' + \frac{(jh)^3}{3!} f_n'' + \frac{(jh)^4}{4!} f_n''' + \frac{(jh)^5}{5!} f_n^{iv} + \frac{(jh)^6}{6!} f_n^{v}$$

$$y_{n+j}''' \equiv y'''(t_n + jh) \cong f_n + jhf_n' + \frac{(jh)^2}{2!} f_n'' + \frac{(jh)^3}{3!} f_n''' + \frac{(jh)^4}{4!} f_n^{iv} + \frac{(jh)^5}{5!} f_n^{v} + \frac{(jh)^6}{6!} f_n^{vi}$$

We proceed with the implementation by substituting the known values of $t_n$ *and* $y_n$ into the differential equations. Then, the differential equation is differentiated to obtain the expression for higher derivatives via partial differentiation as follows;

$$y''' = f(t, y, y', y'') = f_j$$

$$y^{iv} = f_t + y' f_y + y'' f_{y'} + ff_{y''} = \left( \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + f \frac{\partial}{\partial y''} \right) = Df_j$$

$$\begin{aligned} y^{v} &= f_{tt} + (y')^2 f_{yy} + (y'')^2 f_{y'y'} + f^2 f_{y''y''} + 2y' ff_{ty} + 2y'' f_{ty'} \\ &\quad + 2y'y'' f_{yy'} + 2y' ff_{yy''} + 2y'' ff_{y'y''} + Df_j (f_{y'}) + f_j (y'' + f_{y'}) \\ &= D^2 f_j + (f_{y''}) Df_j + f_j (y'' + f_{y'})_j \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ & D^P f_j \end{aligned}$$

where $p$ is the order of the one-step algorithm. Also, note that

$$D = \left( \frac{\partial}{\partial t} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + f \frac{\partial}{\partial y''} \right) \text{ and } D^2 = D(D)$$

**Results**

***Numerical Experiments***

The one-step algorithm derived in this research shall be employed in finding approximate solutions to third order oscillatory problems of the form (1).

The following notations shall be used in the tables below;

ESJ- Absolute error in Sunday (2018)

ETGS- Absolute error in Taparki, Gurah and Simon (2011)

Exec. $t/\sec$- Execution time per seconds of the newly derived one-step algorithm

**Problem 1:**
Consider the third order oscillatory problem of the form,

$$y'''(t) = -y'(t), \quad y(0) = 0, y'(0) = 1, y''(0) = 2, \quad t \in [0.1, 1] \tag{16}$$

with the exact solution is given by,

$$y(t) = 2(1 - \cos t) + \sin t \tag{17}$$

Source: Sunday (2018)

**Table 1:** Showing the result for Problem 1

| $t$ | Exact Solution | Computed Solution | Error | ESJ | Exec. $t/\sec$ |
|---|---|---|---|---|---|
| 0.1000 | 0.109825086090777 | 0.109825086090777 | 2.4980e-016 | 3.7470e-016 | 0.1513 |
| 0.2000 | 0.238536175112578 | 0.238536175112578 | 4.1633e-016 | 8.3267e-016 | 0.2051 |
| 0.3000 | 0.384847228410128 | 0.384847228410127 | 8.3267e-016 | 1.3878e-015 | 0.4818 |
| 0.4000 | 0.547296354302881 | 0.547296354302880 | 3.3307e-016 | 1.4433e-015 | 0.6415 |
| 0.5000 | 0.724260414823458 | 0.724260414823458 | 4.4409e-016 | 1.5543e-015 | 0.8017 |
| 0.6000 | 0.913971243575679 | 0.913971243575679 | 1.1102e-016 | 1.9984e-015 | 0.8555 |
| 0.7000 | 1.114533312668715 | 1.114533312668715 | 4.4409e-016 | 2.8866e-015 | 1.2963 |
| 0.8000 | 1.323942672205193 | 1.323942672205192 | 1.3323e-015 | 4.4409e-015 | 1.3709 |
| 0.9000 | 1.540106973086156 | 1.540106973086155 | 4.4409e-016 | 3.5527e-015 | 1.4627 |
| 1.0000 | 1.760866373071619 | 1.760866373071616 | 2.2204e-015 | 5.3291e-015 | 1.6621 |

**Problem 2:**
Consider the third order oscillatory problem of the form,

$$y'''(t) = y''(t) - y'(t) + y(t), \quad y(0) = 1, y'(0) = 0, y''(0) = -1, \ h = 0.01, \ t \in [0.01, 0.05] \tag{18}$$

with the exact solution is given by,

$$y(t) = \cos t \tag{19}$$

Source: Sunday (2018)

**Table 2:** Showing the result for Problem 2

| $t$ | Exact Solution | Computed Solution | Error | ESJ | Exec. $t/\sec$ |
|---|---|---|---|---|---|
| 0.0100 | 0.999950399610039 | 0.999950399610039 | 0.0000e+000 | 1.1102e-016 | 0.3648 |
| 0.0200 | 0.999800805813405 | 0.999800805813405 | 3.3307e-016 | 1.3323e-015 | 0.4203 |
| 0.0300 | 0.999550633459082 | 0.999550633459082 | 0.0000e+000 | 9.6589e-015 | 1.6058 |
| 0.0400 | 0.999200106660978 | 0.999200106660978 | 0.0000e+000 | 3.2974e-014 | 1.7975 |
| 0.0500 | 0.998750260394966 | 0.998750260394968 | 1.3323e-015 | 8.2379e-014 | 1.9402 |

**Problem 3:**
Consider the third order oscillatory problem of the form,

$$y'''(t) = 3\sin t, \quad y(0) = 1, y'(0) = 0, y''(0) = -2, \quad t \in [0.1, 1] \tag{20}$$

with the exact solution is given by,

$$y(t) = 3\cos t + \frac{t^2}{2} - 2 \tag{21}$$

Source: Sunday (2018)

**Table 3:** Showing the result for Problem 3

| $t$ | Exact Solution | Computed Solution | Error | ESJ | Exec. $t/\sec$ |
|---|---|---|---|---|---|
| 0.1000 | 0.990012495834077 | 0.990012495834077 | 3.3307e-016 | 4.6185e-014 | 0.0427 |
| 0.2000 | 0.960199733523725 | 0.960199733523725 | 3.3307e-016 | 1.8563e-013 | 0.0482 |
| 0.3000 | 0.911009467376818 | 0.911009467376818 | 3.3307e-016 | 4.1578e-013 | 0.0537 |
| 0.4000 | 0.843182982008655 | 0.843182982008655 | 1.1102e-016 | 7.3574e-013 | 0.0592 |
| 0.5000 | 0.757747685671118 | 0.757747685671118 | 1.1102e-016 | 1.1424e-012 | 0.0647 |
| 0.6000 | 0.656006844729034 | 0.656006844729035 | 4.4409e-016 | 1.6327e-012 | 0.0701 |
| 0.7000 | 0.539526561853465 | 0.539526561853465 | 5.5511e-016 | 2.2020e-012 | 0.1481 |
| 0.8000 | 0.410120128041496 | 0.410120128041496 | 5.5511e-016 | 2.8458e-012 | 0.2267 |
| 0.9000 | 0.269829904811993 | 0.269829904811993 | 7.2164e-016 | 3.5596e-012 | 0.4018 |
| 1.0000 | 0.120906917604418 | 0.120906917604419 | 1.0547e-015 | 4.3369e-012 | 0.4075 |

**Problem 4:**
Consider the third order oscillatory problem of the form,

$$y'''(t) = -4y'(t) + t, \quad y(0) = y'(0) = 0, y''(0) = 1, \quad t \in [0.1,1] \tag{22}$$

with the exact solution is given by,

$$y(t) = \left(\frac{3}{16}\right)(1 - \cos 2t) + \left(\frac{1}{8}\right)t^2 \tag{23}$$

Source: Sunday (2018)

**Table 4:** Showing the result for Problem 4

| $t$ | Exact Solution | Computed Solution | Error | ESJ | Exec. $t/\sec$ |
|---|---|---|---|---|---|
| 0.1000 | 0.004987516654767 | 0.004987516654768 | 1.2854e-015 | 8.3209e-013 | 0.2015 |
| 0.2000 | 0.019801063624459 | 0.019801063624469 | 9.5410e-015 | 3.4752e-012 | 0.2074 |
| 0.3000 | 0.043999572204435 | 0.043999572204466 | 3.0732e-014 | 7.8178e-012 | 0.2136 |
| 0.4000 | 0.076867491997407 | 0.076867491997476 | 6.9889e-014 | 1.3681e-011 | 0.2195 |
| 0.5000 | 0.117443317649724 | 0.117443317649855 | 1.3059e-013 | 2.0825e-011 | 0.2254 |
| 0.6000 | 0.164557921035624 | 0.164557921035838 | 2.1452e-013 | 2.8962e-011 | 0.2313 |
| 0.7000 | 0.216881160706205 | 0.216881160706527 | 3.2155e-013 | 3.7764e-011 | 0.2371 |
| 0.8000 | 0.272974910431492 | 0.272974910431941 | 4.4914e-013 | 4.6879e-011 | 0.2430 |
| 0.9000 | 0.331350392754954 | 0.331350392755547 | 5.9286e-013 | 5.5941e-011 | 0.2488 |
| 1.0000 | 0.390527531852590 | 0.390527531853336 | 7.4624e-013 | 6.4592e-011 | 0.2546 |

**Problem 5:**
Consider the third order oscillatory problem of the form,

$$y'''(t) = \cos t, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 2, \quad t \in [0.1,1] \tag{24}$$

with the exact solution is given by,

$$y(t) = t^2 + 3t + 1 - 3\sin t \tag{25}$$

Source: Taparki, Gurah and Simon (2011)

**Table 5:** Showing the result for Problem 5

| $t$ | Exact Solution | Computed Solution | Error | ETGS | Exec. $t/\sec$ |
|---|---|---|---|---|---|
| 0.1000 | 1.010499750059516 | 1.010499750059516 | 2.2204e-016 | 2.4800000 e-007 | 0.0374 |
| 0.2000 | 1.043992007614816 | 1.043992007614818 | 1.1102e-015 | 7.3740000 e-006 | 0.0433 |
| 0.3000 | 1.103439380015981 | 1.103439380015982 | 4.4409e-016 | 6.0542000 e-005 | 0.0487 |
| 0.4000 | 1.191744973074049 | 1.191744973074050 | 6.6613e-016 | 2.5478700 e-004 | 0.1264 |
| 0.5000 | 1.311723384187391 | 1.311723384187391 | 2.2204e-016 | 7.7601600 e-004 | 0.1725 |
| 0.6000 | 1.466072579814895 | 1.466072579814894 | 1.3323e-015 | 1.9261250 e-003 | 0.2828 |
| 0.7000 | 1.657346938286928 | 1.657346938286926 | 2.4425e-015 | 4.1505400 e-003 | 0.3837 |
| 0.8000 | 1.887931727301432 | 1.887931727301430 | 2.4425e-015 | 8.3637340 e-003 | 0.4474 |
| 0.9000 | 2.160019271117552 | 2.160019271117549 | 3.5527e-015 | 1.4773750 e-002 | 0.5393 |
| 1.0000 | 2.475587045576313 | 2.475587045576309 | 3.9968e-015 | 2.4701998 e-002 | 0.5447 |

**Discussion of Result**

The results obtained in Tables 1-5 clearly show that the one-step algorithm derived is computationally reliable and efficient. This is because the computed solution matches the exact solution. In fact, the method obviously performed better than the ones with which we compared our results. The algorithm is also efficient because from the tables, the execution times per seconds are very small. This shows that the algorithm generates results very fast. Thus, there is economy of time in the computation.

**Conclusion**

A computationally reliable one-step algorithm for the solution of third order oscillatory problems of the form (1) has been derived in this research. The results obtained on the application of the algorithm show that it is highly efficient. The paper also analyzed some basic properties of the algorithm which include order, convergence, consistence, zero-stability and region of absolute stability. This analysis further buttresses the fact that the newly derived one-step algorithm can handle the differential equations for which it was designed.

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