Adamawa State University Journal of Scientific Research
Volume 7 Number 1, April, 2019; Article no. ADSUJSR 0701004
ISSN:2705-1900(Online); ISSN: 2251-0702 (Print)
http://www.adsujsr.com

An Alternative Computational Approach for the Simulation of Autonomous Dynamical Differential Equations with One, Two and Three Fixed Points

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Abstract

Autonomous Dynamical Differential Equations (ADDEs) with one, two and three fixed points have been found to be applicable in various fields of human endeavor. There is therefore need to find approximate solutions to such differential equations since some of them do not have solutions in closed form. In view of this, we are motivated to develop an alternative computational approach called the Non-Standard Finite Difference Method (NSFDM) for the simulation of such problems. In developing this alternative computational approach, three major steps were adopted. These include the reconstruction of the numerator function, the reconstruction of the denominator function and the nonlocal representation of nonlinear terms that may occur in the ADDE. The research went further to analyze the basic properties of the NSFDM which include positivity of solutions, elementary stability, dynamical consistence, monotone dependence on initial value and monotonicity of solutions. Finally the ADDEs with one, two and three fixed-points were simulated using the new approach in order to test its computational reliability.

Keywords: ADDEs; dynamical systems; fixed points; NSFDM; simulation
AMS Subject Classification: 65L05, 65L06, 65D30

Introduction

Over the recent decades, many physical phenomena have been modeled using ADDEs. These equations have received some remarkable attention due to its classical applications in sciences and engineering. According to Erdi (2015), one of the most important reasons for using NSFDM is to be able construct discrete models which have correct qualitative behavior with the corresponding differential equation. Although there is no general procedure to achieve this point, there have been some powerful results for some types of differential equation. For instance when considering first order ADDE, numerical instabilities occur in discrete modeling if the linear stability properties of any fixed-points of difference equation is not in concordance with those of the corresponding differential equation.

Autonomous dynamical systems are mainly represented by a state that evolves in time. A dynamical system/equation is a system/equation in which a function describes the time dependence of a point in a geometrical space. Examples include mathematical models that describe growth, decay, swinging of a clock pendulum, the flow of water in a pipe, the number of fish each time in a lake, etc. The input as well as the current state of a dynamical system determines the evolution of the system. An important characteristic of a dynamical system is whether it is continuous or discrete. Continuous systems (often called flows) are given by differential equations whereas discrete systems (often called maps) are specified by difference equations. This research however will focus on the latter.

A highly efficient computational approach (NSFDM) shall be formulated for the simulation of ADDE of the form,

$$\frac{dy}{dt} = f(y),\ y(t_0) = y_0 \tag{1}$$

where $y = \left[y^1, y^2, ..., y^n\right]^T : \left[t_0, T\right) \rightarrow \Re^n$, $f = \left[f^1, f^2, ..., f^n\right]^T : \Re^n \rightarrow \Re^n$ is differentiable and $t_0 \in \Re^n$, $\Re^n$ is the set of n-tuples, where n-tuple is a sequence (or finite ordered list) of n elements, where n is a non-negative integer.

We also assume that the ADDE (1) satisfies the property below which guarantees the existence and uniqueness of its solution.

The property states thus; a function $f:\Re^n \to \Re^n$ is said to be Lipschitz on $B \subset \Re^n$ with Lipschitz constant $L \geq 0$ if,

$$\left| f(x) - f(y) \right| \leq L\left| x - y \right| , \ \forall x, y \in B \quad (2)$$

If f is Lipschitz on $\Re^n$, then f is said to be globally Lipschitz. If f is Lipschitz on every bounded subset of $\Re^n$, then f is said to be locally Lipschitz. The concept of Lipschitzian functions is important in the proof of existence and uniqueness results for many problems in mathematics. The theorem below follows from this definition.

Theorem 1 (Kama, 2009)
Let $f:\Re^n \to \Re^n$ be globally Lipschitz. Then, there exist a unique solution $y(t)$ to (1) for all $t \geq 0$. Hence, equation (1) defines an ADDE on $\Re^n$.
See, Kama (2009) for proof.
Let $F:\Re^n \to \Re^n$. Consider a sequence $\{y_k\}_{k=0}^{\infty}$ defined recursively by,

$$y_{k+1} = F(y_k) \quad (3)$$

We refer to such a map or iterate as explicit mapping since y_{k+1} is given explicitly in terms of y_k. Sometimes y_{k+1} is not given by an explicit mapping of the form (3), but instead y_{k+1} is obtained from y_k through an implicit mapping of the form,

$$G(y_{k+1}, y_k) = 0 \quad (4)$$

where $G:\Re^n \times \Re^n \to \Re^n$.

Note that for (3), uniqueness of the solution sequence $\{y_k\}$ is guaranteed due to the explicit nature of the map, whereas for (4) it is necessary to establish existence and uniqueness of a solution y_{k+1} when y_k is given.
According to Mickens (1994), the general form of NSFDM can be written as,

$$y_{k+1} = F(h, y_k) \quad (5)$$

Note that equation (3) is of the form (5).

Definition 2 (Borowski and Borwein, 2005)
A differential equation is autonomous if it does not depend on the variable of differentiation (often time), that is, such that there is no explicit occurrence of the independent variable in the equation.

Definition 3 (Simulation)
Simulation is the use of mathematical model to recreate a situation, often repeatedly, so that the likelihood of various outcomes can be more accurately estimated. It is also the imitation of the operation of real-world process or system over time. The behavior of a system that evolves over time is studied by developing a simulation model. Source: *wikipedia.org/wiki/Simulation.*

Definition 4 (Anguelov and Lubuma, 2003)
The fixed-point (equilibrium) of the differential equation (1) is any zero $\bar{y}$ of the function $f : f(\bar{y}) = 0$.

Definition 5 (Anguelov and Lubuma, 2001)
A finite difference scheme is called NSFDM, if at least one of the following conditions is satisfied;

i) in the discrete derivative, the traditional denominator is replaced by a non-negative function φ such that,

$$\varphi(h) = h + O(h^2), \ as \ h \to 0 \quad (6)$$

ii) non-linear terms that occur in the differential equation are approximated in a non-local way i.e. by a suitable functions of several points of the mesh. For example

$$y^2 \approx y_k y_{k+1}, y_{k-1} y_k, \ y^3 \approx y_{k-1} y_k y_{k+1}, \ y_k^2 y_{k+1}$$

A lot of authors have proposed different NSFDMs for the simulation of ADDE of the form (1), see the works of Sunday (2010), Sunday, Ibijola and Skwame (2011), Liu, Clemence and Mickens (2011), Sunday, James and Bakari (2015), Wood (2015), Garba *et al.* (2015), to mention a few.

Formulation of the Alternative Computational Approach (NSFDM)
We shall formulate an alternative computational approach called the NSFDM for the simulation of ADDEs with one, two and three fixed points. This alternative computational approach (which is a difference equation) is meant to have the same qualitative properties as the corresponding differential equation. According to Patidar (2005), the construction of NSFDM is not always straight forward and there are no general criteria for doing so. Thus, the alternative computational approach shall be

formulated by modifying the numerator and denominator functions to suit the qualitative behavior of the corresponding differential equations by assuming that,

$$\frac{dy}{dt} \approx \frac{y_{k+1} - \psi(h) y_k}{\varphi(h)} \qquad (7)$$

where $\psi(h)$ *and* $\varphi(h)$ (functions of the step-size $h = \Delta t$) are respectively the numerator and denominator functions, $t_k = t_0 + hk$ *and* $y(t) = y_k$. The $\psi(h)$ *and* $\varphi(h)$ have the following properties,

$$\left.\begin{array}{l} \psi(h) = 1 + O(h) \\ \varphi(h) = h + O(h^2) \end{array}\right\} \qquad (8)$$

The conventional discrete representation for the first derivative takes $\psi = 1$ *and* $\varphi = h$ for standard finite difference methods as $h \to 0$. It therefore implies that there should be a systematic way for constructing a denomination function for a NSFDM. Also, unless the differential equation has 'dissipation', the numerator function is usually equal to one. This has also been verified by Mickens (2005).

Suppose the fixed-points of (1) is given by,

$$\left\{\bar{y}^{(i)},\ i = 1,2,...,n\right\}$$

where n may be bounded. The fixed-points are the real n solutions to the equations,

$$f\left(\bar{y}\right) = 0 \qquad (9)$$

Let P_i be defined by,

$$P_i = \left.\frac{df}{dy}\right|_{y=\bar{y}^{(i)}} \qquad (10)$$

and P^* as,

$$P^* \equiv Max\left\{\left|P_i\right|, i = 1,2,...,n\right\} \qquad (11)$$

The alternative computational approach that approximates the ADDES of the form (1) is given by the expression,

$$\frac{y_{k+1} - \psi(h) y_k}{\varphi(h)} = f(y_k) \qquad (12)$$

where the denominator function is modified as,

$$\varphi(h) = \frac{\varphi(h, P^*)}{P^*} \qquad (13)$$

This form replaces the simple $'h'$ function found in the standard finite difference method,

$$\frac{dy}{dt} \to \frac{y_{k+1} - y_k}{h} \qquad (14)$$

Note that $\varphi(h, P^*)$ in equation (13) has the properties

$$\varphi(h, P^*) = h + O(P^*, h^2),\ 0 < \varphi < \frac{1}{P^*} \qquad (15)$$

If we consider an autonomous dynamical system where the independent variable t is time, it follows that P_i have units of inverse time and a set of time scales can be defined by means of the relations,

$$T_i = \frac{1}{P_i},\ i = 1,2,...,n;\ T^* = \frac{1}{P^*} \qquad (16)$$

Thus, T^* corresponds to the smallest time scale and this shows that the denominator function is in the range $0 < \varphi(h, T^*) < T^*$.

Therefore, substituting equation (13) into (12), we obtain the alternative computational approach as,

$$\frac{y_{k+!} - \psi(h) y_k}{\left(\frac{\varphi(h, P^*)}{P^*}\right)} = f(y_k)$$

That is,

$$y_{k+1} = \psi(h) y_k + \left(\frac{\varphi(h, P^*)}{P^*}\right) f(y_k) \qquad (17)$$

Equation (17) is the alternative computational approach called the NSFDM that gives correct linear stability property for any ADDE of the form (1).

Analysis of Basic Properties of the Alternative Computational Approach (NSFDM)

The analysis of basic properties of the alternative computational approach formulated shall be carried out in this section. These properties include positivity of solutions, elementary stability, dynamical consistence, monotone dependence on initial value and monotonicity of solutions.

Let us assume that the function $F(h, y)$ in equation (5) has a continuous derivative with respect to both variables for $h > 0, y \in \Re$ and satisfies,

$$\left.\begin{array}{l} F(0, y) = y \\ \dfrac{\partial F(0, y)}{\partial h} = f(y) \end{array}\right\} \qquad (18)$$

Another assumption made is that the difference scheme (5) is consistent with the ADDE (1). We note that consistency implies that (18) is satisfied when y is the solution of the ADDE (1).

Positivity of Solutions

Definition 6 (Wood, 2015)
The NSFDM (5) is called positive if, for any value of the step-size h and $t_0 \in \Re_+^n$, its solution remains positive, that is $y_k \in \Re_+^n \quad for \ \ all \ \ k \in \aleph$.

Positivity of solution is an important property in applications to biological systems and various physical systems where negative values are generally not meaningful. This property has been satisfied based on the trajectory of the graphical results that shall be presented in Figures 1, 2 and 3 below.

Elementary Stability

Definition 7 (Wood, 2015)
The NSFDM (5) is called elementary stable if, for any value of the step-size h, its only fixed-points $\bar{y}$ are the same as the equilibria of the differential system (1) and the local stability properties of each $\bar{y}$ are the same for both the differential system and the discrete method.
This fundamental property has also been satisfied by the alternative computational approach (17).

Dynamical Consistence

Definition 8 (Wood, 2015)
The NSFDM (5) is said to be dynamically consistent with the ADDE (1) if it is both positive and elementary stable.
Thus, the alternative computational approach (17) is dynamically consistent.

Theorem 2 (Anguelov and Lubuma, 2003)
The NSFDM (5) is stable with respect to monotone dependence on initial value if,

$$\frac{\partial F(h, y_k)}{\partial y} \geq 0, \ y \in \Re, \ h > 0 \qquad (19)$$

It is important to note that the alternative computational approach (17) satisfies Theorem 2.
Proof

Let the numerator function ψ and the denominator function $\frac{\varphi(h, P^*)}{P^*}$ of the alternative computational approach (17) be defined by,

$$\left.\begin{aligned} \psi &= 1 \\ \frac{\varphi(h, P^*)}{P^*} &= \varphi \end{aligned}\right\} \qquad (20)$$

Substituting (20) in (17), gives

$$y_{k+1} = y_k + \varphi f(y_k) \qquad (21)$$

Now that (21) is of the form (5), we have

$$F(h, y_k) = y_k + \varphi f(y_k) \qquad (22)$$

Differentiating equation (22) partially with respect to y gives,

$$\frac{\partial F}{\partial y} = \frac{\partial (y)}{\partial y} + \frac{\partial (\varphi)}{\partial y} f(y) + \frac{\varphi \partial [f(y)]}{\partial y} \qquad (23)$$

For all $y \in \Re$ and $h > 0$, equation (23) satisfies (19). This therefore shows that the alternative computational approach (17) is stable with respect to monotone dependence on initial value.

Monotonicity of Solutions

Due to the autonomous nature of the differential equation (1), its solution has a relatively simple structure with regard to their monotonicity. Every solution is either increasing or decreasing on the whole interval $[t_0, \infty)$. The increasing and decreasing solutions are separated by fixed-points.

Definition 9 (Anguelov and Lubuma, 2003)
The NSFDM (5) is stable with respect to the property of monotonicity of solutions if for all $y_0 \in \Re$, the solution y_k of the NSFDM (5) is increasing or decreasing whenever the solution $y(t)$ of the ADDE (1) is increasing or decreasing. This property has been satisfied by the alternative computational approach (17) in view of the graphical results obtained.

Results

We shall apply the alternative approach (NSFDM) developed in simulating some ADDE with one, two and three fixed points. Consider the problems below:

Problem 1 *(ADDE with One Fixed-Point: The Growth Model)*

A bacteria culture is known to grow at a rate proportional to the amount present. After one hour, 1000 strands of the bacteria are observed in the culture; and after four hours, 3000 strands. Find the number of strands of the bacteria present in the culture at time $t: 0 \le t \le 1$.

Let $y(t)$ denote the number of bacteria strands in the culture at time t, the initial value problem modeling this problem is given by,

$$\frac{dy}{dt} = 0.366y,\ y(0) = 694 \tag{24}$$

The exact solution is given by,

$$y(t) = 694\ e^{0.366t} \tag{25}$$

Source: Sunday, Yusuf and Andest (2016)

Comparing (24) with (1), we see that

$$f(y) = 0.366y \tag{26}$$

and $\overline{y}^{(1)} = 0$ is the only fixed-point. Then, on the application of (10), we obtain

$$P_1 = \left.\frac{df}{dy}\right|_{y=\overline{y}^{(1)}=0} = 0.366 \tag{27}$$

From equation (11),

$$P^* = 0.366 \tag{28}$$

The numerator function ψ and the denominator function φ for equation (24) are defined by,

$$\left.\begin{aligned} \psi &= 1 \\ \frac{\varphi(hP^*)}{P^*} &= \frac{1-e^{0.366h}}{0.366} \end{aligned}\right\} \tag{29}$$

Substituting (29) into (17), gives

$$y_{k+1} = y_k + \left(\frac{1-e^{0.366h}}{0.366}\right)(0.366)y_k \tag{30}$$

Equation (30) is the alternative computational approach/NSFDM for the autonomous dynamical growth model (24). Equation (30) is of the form (5).

Simulating Problem 1 using the newly formulated alternative computational approach/NSFDM, we obtain the graphical result presented in Figure 1. The simulation result is compared with the exact solution of the problem.

Problem 2 *(ADDE with Two Fixed-Points: The Logistic Model)*

The logistic model (an extension of growth model) is the law that regulates with good approximation the growth rate of a certain population number as function of time. The model is based on the assumption that the population evolves in an environment with limited resources with no immigration or emigration phenomena. Let $x(t)$ be the population size at time t, the law that regulates it can be expressed by the first-order ADDE,

$$\frac{dx}{dT} = rx\left(1-\frac{x}{k}\right) \tag{31}$$

where $k > 0$ is the carrying capacity of the system/environment, $r > 0$ is a parameter called intrinsic growth rate ($r = b - d$, where b is the instantaneous birth rate and d the instantaneous death rate).

We therefore carry out non-dimensionalization (scaling) of equation (31). Since the model in (31) has four parameters, we reduce the number of parameters by scaling as follows. Let,

$$T = \frac{t}{r}\ and\ x = ky \tag{32}$$

Substituting equation (32) in (31), gives

$$\frac{d(ky)}{d\left(\frac{t}{r}\right)} = rky\left(1-\frac{ky}{k}\right) \tag{33}$$

which reduces to,

$$\frac{dy}{dt} = y(1-y) \tag{34}$$

Equation (34) is called the logistic model. The equation can be solved by the method of separation of variable to give the exact solution,

$$y(t) = \frac{y_0}{y_0 + (1-y_0)e^{-t}} \tag{35}$$

where, $y(t_0) = y_0$.

It is important to state that this equation possess a very simple asymptotic dynamics: all solutions with positive initial condition ($y_0 > 0$) will eventually approach the carrying capacity k. Therefore, population size will eventually be stabilized to k in the long run if population dynamics initially either overshoot or undershoot the carrying capacity.

Here, we shall consider a special case of equation (34) given by substituting $r = k = 1$ in equation (31). This gives,

$$\frac{dy}{dt} = y(1-y), y(0) = 0.5 \tag{36}$$

with the theoretical solution,

$$y(t) = \frac{0.5}{0.5(1+e^{-t})} \quad (37)$$

Source: Sunday, Yusuf and Andest (2016)
Comparing (36) with (1), we obtain

$$f(y) = y(1-y) \quad (38)$$

The equation has two fixed points given by,

$$\left.\begin{array}{l} \bar{y}^{(1)} = 0 \\ \bar{y}^{(2)} = 1 \end{array}\right\} \quad (39)$$

On the application of (10), we obtain

$$\left.\begin{array}{l} P_1 = \left.\frac{df}{dy}\right|_{y=\bar{y}^{(1)}=0} = 1 \\ P_2 = \left.\frac{df}{dy}\right|_{y=\bar{y}^{(1)}=1} = -1 \end{array}\right\} \quad (40)$$

From equation (11),

$$P^* = 1 \quad (41)$$

The numerator function ψ and the denominator function φ for equation (36) are defined by,

$$\left.\begin{array}{l} \psi = 1 \\ \frac{\varphi(hP^*)}{P^*} = \frac{1-e^{-(1)h}}{1} \end{array}\right\} \quad (42)$$

Substituting (42) into (17), gives

$$y_{k+1} = y_k\left[1+(1-e^{-h})(1-y_k)\right] \quad (43)$$

Equation (43) is the alternative computational approach/NSFDM for the autonomous dynamical logistic model (36). Equation (43) is of the form (5). Simulating Problem 2 using the newly formulated alternative computational approach/NSFDM, we obtain the graphical result presented in Figure 2. The simulation result is compared with the exact solution of the problem.

Remark: we can see from the Figure 2 that the curve is asymptotic about the carrying capacity $k=1$.

Problem 3 *(ADDE with Three Fixed-Points: The Combustion Model)*
Consider the ADDE with three fixed points given by,

$$\frac{dy}{dt} = y(1-y^2),\ y(0) = 0.5 \quad (44)$$

Equation (44) is an elementary model for combustion. According to Kama (2009), despite the simple nature of (44), its solution cannot be written in a closed form.
Comparing (44) with (1), we obtain

$$f(y) = y(1-y^2) \quad (45)$$

The equation has three fixed points given by,

$$\left.\begin{array}{l} \bar{y}^{(1)} = 0 \\ \bar{y}^{(2)} = \sqrt{1} \\ \bar{y}^{(3)} = -\sqrt{1} \end{array}\right\} \quad (46)$$

On the application of (10), we obtain

$$\left.\begin{array}{l} P_1 = \left.\frac{df}{dy}\right|_{y=\bar{y}^{(1)}=0} = 1 \\ P_2 = \left.\frac{df}{dy}\right|_{y=\bar{y}^{(2)}=\sqrt{1}} = -2 \\ P_3 = \left.\frac{df}{dy}\right|_{y=\bar{y}^{(3)}=-\sqrt{1}} = -2 \end{array}\right\} \quad (47)$$

From equation (11),

$$P^* = 2 \quad (48)$$

The numerator function ψ and the denominator function φ for equation (44) are defined by,

$$\left.\begin{array}{l} \psi = 1 \\ \frac{\varphi(hP^*)}{P^*} = \frac{1-e^{-2h}}{2} \end{array}\right\} \quad (49)$$

Substituting (49) into (17), gives

$$y_{k+1} = y_k + \left(\frac{1-e^{-2h}}{2}\right)y_k(-y_k y_{k+1}) \quad (50)$$

Equation (50) is the alternative computational approach/NSFDM for the autonomous combustion model (44). Equation (50) is of the form (5).

Simulating Problem 3 using the newly formulated alternative computational approach/NSFDM, we obtain the graphical result presented in Figure 3. The simulation result is compared with that of Sunday *et. al.* (2016) since the combustion model does not have a closed form (exact) solution.

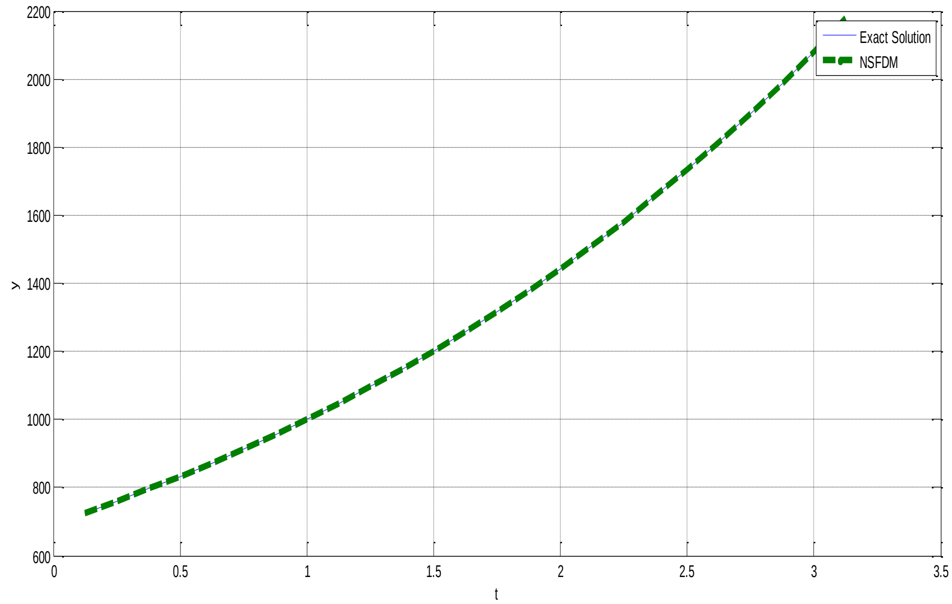

Figure 1: Graphical Result for Problem 1

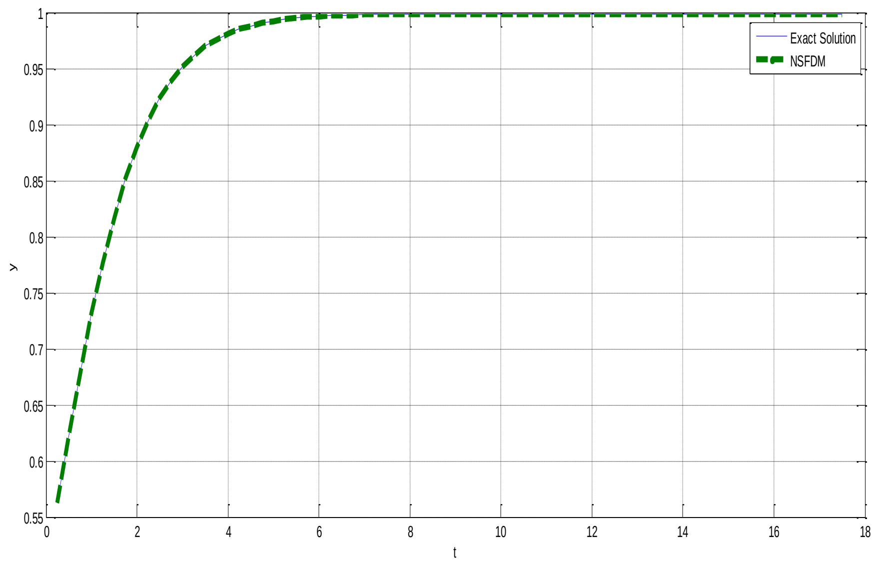

Figure 2: Graphical Result for Problem 2

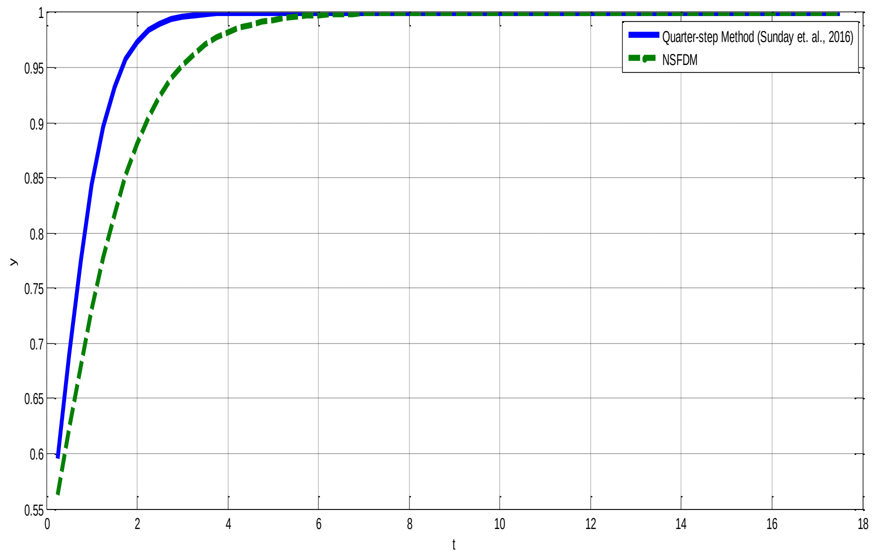

Figure 3: Graphical Result for Problem 3

Discussion of Results

From the graphical results presented Figures 1, 2 and 3 (for Problems 1, 2 and 3 respectively), it is obvious that the alternative computational approach (i.e. the NSFDM) is computationally reliable. This is because from the simulation result, it is clear that the method effectively approximates the ADDEs.

Conclusion

The computational approach/NSFDM adopted in this research has been shown to effectively simulate ADDEs with one, two and three fixed points. The need for this approach came up due to some shortcomings of existing methods in which the qualitative properties of the exact solutions are not usually transferred to the numerical (approximate) solutions. The paper went further to analyze the basic properties of the method which include positivity of solutions, elementary stability, dynamical consistence, monotone dependence on initial value and monotonicity of solutions.

References

Anguelov, R. & Lubuma, J. M. S. (2001). Contributions to the mathematics of the nonstandard finite difference method and applications. *Numerical Methods for Partial Differential Equations*, 17(5):518–543.

Anguelov, R. & Lubuma, J. M. S. (2003). Non-Standard Finite Difference Method by Non-Local Approximation. *Mathematics and Computer in Simulation*, 61:465-475.

Borowski, E. J. & Borwein, J. M. (2005). *Dictionary of Mathematics.* Harper Collins Publishers, Glasgow.

Erdi, K. (2015). *Solution of Differential Equations using Nonstandard Finite Difference Methods.* An M.Sc. Thesis Submitted to the Institute of Science of Hacettepe University, Ankara-Turkey.

Garba, S., Gumel, A., Hassan, A. & Lubuma, J. M-S. (2015). Switching from exact scheme to nonstandard finite difference scheme for linear delay differential equation. *Applied Mathematics and Computation*, 258:388–403.

Kama, P. (2009). *Nonstandard finite difference methods in dynamical systems.* A PhD Thesis Submitted to the Department of Mathematics and Applied Mathematics, University of Pretoria-South Africa.

Liu, L., Clemence, D. P. & Mickens, R. E. (2011). A nonstandard finite difference scheme for contaminant transport with kinetic Langmuir sorption. *Numerical Methods for Partial Differential Equations*, 27(4):767–785.

Mickens, R. E. (1994). *Non-Standard Finite Difference Models of Differential Equations.* World Scientific, Singapore.

Mickens, R. E. (2005). Dynamic consistency: a fundamental principle for constructing nonstandard finite difference schemes for differential equations. *Journal of*

Difference Equations and Applications, 11(7):645–653.

Patidar, K. C. (2005). On the use of nonstandard finite difference methods. *Journal of Difference Equations and Applications*, 11(8):735–758.

Sunday, J. (2010). On Exact Finite Difference Scheme for Numerical Solution of Initial Value Problems in ODEs. *Pacific Journal of Science and Technology (A Publication of Akamai University, USA)*, 11(2): 260-267.

Sunday, J., Ibijola, E. A. & Skwame, Y. (2011). On the Theory and Applications of Nonstandard Finite Difference Method for Singular ODEs. *Journal of Emerging Trends in Engineering and Applied Sciences,* 2(4): 684-688.

Sunday, J., James, A. A. & Bakari A. I. (2015). A New Non-Standard Finite Difference Method for Autonomous Differential Equations. *Engineering Mathematics Letters*, 9, 1-14.

Sunday, J., Yusuf, D. & Andest, J. N. (2016). A quarter-step computational hybrid block method for first-order modeled differential equations using Laguerre polynomial. *Engineering Mathematics Letters*, 4:1-16.

Wood, D. (2015). *Advancements and Applications of NSFDMs.* A PhD Thesis Submitted to the Faculty of Graduate School, University of Texas-Arlington.