Adamawa State University Journal of Scientific Research
Volume 7 Number 2, August, 2019; Article no. ADSUJSR 0702031
ISSN:2705-1900(Online); ISSN: 2251-0702 (Print)
http://www.adsujsr.com

Characterization of Signed Symmetric Group in Inner Product Spaces

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Abstract

This paper provides some characterizations of signed symmetric group (SS_n) using the notion of orthogonality in inner product spaces. The concepts of ortho-stochastic and reflection were introduced on SS_n and results were established with some examples .

Keywords: Signed symmetric group, Ortho-stochastic, Trace, Contraction, Orthogonal, Orthonormal, Reflection.

Introduction

The origins of a group theory are in the study of permutations and the symmetric groups. The group of all permutationns of a set is an object of importance within the abstract study. Full transformation semigroup is also known as full symmetric semigroup or a permutation of a set X_n which is a bijection function $\alpha : X_n \to X_n$. The group of permutation of set X_n denoted by S_n is called the symmetric group of n which has $n!$ elements that is $|S| = n!$. If n is some positive integers, we can consider the set of all $n \times n$ matrix over the real value. This is a group with matrix multiplication which is called the General Linear group (GL_n) and defined as $GL_n = \{n \times n \ matrices \ \ A \ with \ detA \ = \ 0\}$.

The matrices with determinant 1 is called the Special Linear group (SLn) and defined as $SL_n = \{n \times n \ matrices \ \ B \ with \ detB \ = \ 1\}$.

The orthogonal group of dimensions n denoted by GO_n is the group of $n \times n$ orthogonal matrices where the group operation is given by matrix multiplication and an orthogonal matrix is a real matrix whose inverses equals its transpose. The determinant of orthogonal matrix is either 1 or -1 and the trace of matrix A written as $tr(A)$ is defined as the sum of the diagonal elements.

In (2015), Mogbonju studied the combinatorial properties of signed transformation semigroups and analysed its structure in matrix notation. Richard (2008) also defined permutation matrices with a permutation α of X_n and the associated $n \times n$ permutation matrix $\prod(\alpha)$ as

$$\prod_{ij}(\alpha) = \begin{cases} 1, & if \ \ \alpha(j) \ = i \\ 0, & otherwise \end{cases}$$

For example, let $\alpha \subset SS_4$ be defined as $\alpha = (2 \ 1 \ -4](4 \ 3 \ -2]$, we then placed ± 1 in the (i, j) - entry to indicate $j \to \pm i$ which can be written in matrix as:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

In linear algebra, an inner product space is a vector space whose additional structure associates each pair of vectors in the space with a scalar quantity which also provide the means of orthogonality.

The concept of ortho-vector, stochastic vector and reflection was introduced by Gudder and Latremoliere (2009) in Boolean inner product space. More so, in

(2012), Asit and Madhumangal modified this concepts on fuzzy inner product spaces. For more and recent work on group and semigroup theory, see[Howie (1995), Laradji and Umar (2007), Rauf and Usamot (2017) and Usamot et. al.(2018)].

The goal of this paper is to introduce and extend these concepts of orthogonality, ortho-stochastic and reflection to signed symmetric group in inner product spaces.

Basic Definitions

Definition 1 : Signed Symmetric Group(Mogbonju 2015)

A signed symmetric group denoted by (SS_n) can be defined as the mapping $\alpha : Dom(\alpha) \subseteq X_n \to Im(\alpha) \subset Z_n$ where $X_n = \{1,2,3,...,n\}$ and $Z_n = \{\pm 1, \pm 2, \pm 3,...,\}$.

Definition 2 : Symmetric and Orth-stochastic (Asit and Madhumangal 2014)

A matrix A is said to be invertible if $A^{-1} = A^* = A$. Hence, we say that A is symmetric and orth-stochastic.

Definition 3 : Orth-stochastic (Asit and Madhumangal 2014)

Let $A = [a_{ij}]_{nxn}$ be a matrix on V_n . Then, A is said to be ortho-stochastic matrix if

$A^* A \geq I$ and $AA^* \geq I$.

Definition 4 : Reflection (Asit and Madhumangal 2014)

Matrix A is said to be a reflection if it is symmetric and ortho-stochastic, that is an orthogonal matrix of order 2 .

Lemma 5 : (Asit and Madhumangal 2014)

Matrix A is symmetric ortho-stochastic if and only if A is an orthogonal matrix of order 2 .

Proposition 6: (Asit and Madhumangal 2014)

The sum and product of reflections need not to be a reflection.

Definition 7: Contraction Mapping (Adeshola 2013)

A mapping α for which $|\alpha x - \alpha y| \leq |x - y|$, $\forall\ x, y \in X_n$ is called a contraction.

Definition 8: Orthogonal and Orthonomal set (Kreyszig 1978)

Let a set $S = \{u_1, u_1, ..., u_n\}$ of non zero vectors in an inner product space V , then S is called orthogonal if each pair of vectors in S satisfy $\langle u_i, u_j \rangle = 0$ for $i \neq j$ and S is called orthonomal if S is orthogonal and each vector in S has unit length, that is

$$\langle u_i, u_j \rangle = \begin{cases} 0, & if\ \ i \neq j \\ 1, & for\ \ i = j \end{cases}$$

Main Results

In this section, the concept of orthogonality in inner product spaces were used to obtain some properties of signed symmetric group.

Proposition 1:

Let A, B be any $n \times n$ matrix on SS_n, then

(i) $tr(A) = tr(A^T)$ and $tr(B) = tr(B^T)$

(ii) $tr(AA^T) = tr(BB^T)$

(iii) $tr(AB) = tr(A^T B^T)$

Proof:

$(i) \to (iii)$

Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$. By definition of trace, $tr(A) = \sum_{i=1}^{n}(a_{ii})$.

but $A = A^T$ and $tr(A^T) = \sum_{i=1}^{n}(a_{ii}^*)$. Similarly, $tr(B^T) = \sum_{i=1}^{n}(b_{jj}^*)$.

Then, $tr(AB) = \sum_{ij=1}^{n}(\prod_{ij}(a_{ii}b_{jj})) = \sum_{ij=1}^{n}(\prod_{ij}(a_{ii}^* b_{jj}^*)) = tr(A^T B^T)$

(ii) Suppose $A \neq A^T$ and A is symmetric, then $tr(A) = tr(A^T)$.

This implies, $tr(A^T A) = \sum_{ij=1}^{n}(\prod_{ij}(a_{ii}^{*}a_{ii})) = \sum_{ij=1}^{n}(\prod_{ij}(a_{ii}a_{ii}^{*})) = tr(AA^T)$

Similarly, $tr(B^T B) = tr(BB^T)$. Thus, since A and B are ortho-stochastic, then the result follows immediately.

Lemma 2 : Let CSS_n be a contraction mapping in signed symmetric group. Then, every mapping in CSS_n are of order 2 for $n \geq 2$.

Proof : Suppose $\alpha \in CSS_n$ and $|\alpha x - \alpha y| > 1$ for all $x, y \subset SS_n$ then, α is not a contraction. But if $|\alpha x - \alpha y| \leq 1$ implies that $\alpha \in CSS_n$ and $|CSS_n| = 2$ for all $n \geq 2$. Hence the result.

Lemma 3 : Every permutation on SS_n in matrix notation are orthogonal matrix.

Proof: Let $A = [a_{ij}]_{nxn}$ be a matrix in SS_n and each rows of A represent the vectors u_i such that $A = \{u_1, u_2, ..., u_n\}$. Then by Definition 8, $\langle u_i, u_j \rangle = 0$, for $i \neq j$ and since A is orthogonal, the rows of A form an orthogonal set. Again, let v_i be the columns of A such that $\langle v_i, v_j \rangle = 1$ for $i = j$. Then, the columns of A form an orthonomal set. Now, all $A_i \in SS_n$ which consists of vectors u_i are orthogonal matrix and form a basis of the set $\{u_1, u_2, ..., u_n\}$ in the space SS_n. It was observed that, $\langle u_i, u_j \rangle = 1$, for all $i = j$ and $\langle v_i, v_j \rangle = 0$, for $i \neq j$.

Lemma 4: Let *V* be an inner product space with basis $P = \{u_1, u_2, ..., u_n\}$ where $u_n = Im(\alpha) \in SS_n$ then, Matrix Q is the matrix representation of the inner product on *V* relative to the basis P if

(i) $Q = |a_{ij}|$ where $a_{ij} = \langle u_i, u_j \rangle$;

(ii) Q is symmetric and

(iii) P is orthonomal basis

Theorem 5 : Let $V = SS_n$, $A = \lfloor a_{ij} \rfloor_{n\times n} \subset SS_n$ and Q denotes the matrix representation of an inner product relative to basis $P \in SS_n$, then there exits matrix A with rows u_i in inner product space such that

$$Q_{ij} = \begin{cases} 0, & if \;\; i \neq j \\ 1, & for \;\; i = j \end{cases}$$

that is, $[Q_{ij}] = I$ for all $A_i \subset SS_n$.

Proof: Suppose $A = \lfloor a_{ij} \rfloor_{n\times n}$ with rows u_i and column $v_i \in A$. Then, since $A \in SS_n$ the inner product $\langle u_i, u_j \rangle = \langle u_j, u_i \rangle$ which indicates that, the inner products in A are symmetric. Also, Q can be generated from A by equating $\langle u_i, u_j \rangle$ to their correspondencing values i.e $\langle u_i, u_j \rangle = a_{ij}$ for $i \neq j$ and $\langle u_i, u_i \rangle = a_{ii}$ for $i = j$. Then by Lemma 4, Q depends on the inner product of $A \in SS_n$ and since P is an orthonomal basis, then Q is an identity matrix. Hence, the proof.

Lemma 6: Suppose an $n \times n$ matrix $A \in SS_n$, then A is ortho-stochastic if it is symmetric, orthogonal and $A^T A = AA^T = I$ where A^T is the transpose of A for some $A \neq A^T$.

Theorem 7: The set of the elements in matrix notation of SS_n are ortho-stochastic for $n \geq 2$.

Proof: Let $dom(\alpha) \subseteq X_n \to Im(\alpha) \subset Z_n$ and $\{\alpha_1, \alpha_2, \cdots \alpha_n\}$ be the set of permutation in SS_n. Also let $\{A_i\}_{i=1}^n$ denotes the set of an $n \times n$ matrices notation of $\alpha_n \in SS_n$ such that

$$\alpha_n = \begin{pmatrix} a & b & c & d & . & . & . & n \\ -b & a & n & -d & . & . & . & -c \end{pmatrix}$$

for each $n \geq 1$ and $a < b < c < ...n$ which in matrix notation can be express as

$$A_i = \begin{pmatrix} 0 & 1 & 0 & 0 & . & . & . & . & 0 \\ -1 & 0 & 0 & 0 & . & . & . & . & 0 \\ 0 & 0 & 0 & 0 & . & . & . & . & -1 \\ 0 & 0 & 0 & -1 & . & . & . & . & 0 \\ 0 & 0 & 0 & 0 & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 1 & 0 & . & . & . & . & 0 \end{pmatrix}$$

Furthermore, if A_i is symmetric it implies that $A_i = A_i^T$ for some $A_i \in SS_n$. Furthermore, if $A_i \neq A_i^T$ shows that A_i is not symmetric but by Lemma 3 and 6 the result follows immediately.

Example : Let $\alpha : X_6 \to Z_6$ be the permutation in SS_6 be defined by $\alpha = (1\ 4\ 5\ 3)(2\ -6](6\ -2]$ which is equivalent to matrix A defined as:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} = A^T$$

Now, since $A = A^T$ it follows that

$$AA^T = A^T A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = I$$

For $A \neq A^T$, let us consider $\alpha \in SS_4$ such that $\alpha = (1\ 2\ -3](3\ 4\ 1]$ having matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

But, A does not equal to A^T and,

$$AA^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

Also,

$$A^T A == \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

Hence, the conclusion follows from Theorem 7.

Theorem 8: Let A be any matrix notation in SS_n, then the following statements are equivalent:

(i) A is orthogonal;
(ii) A is normal; and
(iii) A is ortho-stochastic.

Proof: (i) Let $\alpha \in SS_n$ and $A = [a_{ij}]_{n \times n}$ be matrix notation of α. Also, let $P = \{u_1, u_2, ..., u_n\}$ be the basis of A such that $\langle u_1, u_2 \rangle = \langle u_1, u_3 \rangle = \langle u_2, u_3 \rangle = \cdots = \langle u_{n-1}, u_n \rangle = 0$. then, by Definition 8 A is orthogonal.

(ii) Let A^* be the adjoint of A this implies $A = A^*$. Since $A \subset SS_n$, then by (i) A^* is also orthogonal and $A^* A = AA^*$ which implies that A is normal.

(iii) From (i) and (ii), $A = A^*$ and assuming $A \neq A^*$ but $A \in SS_n$, then the conclusion follows from Theorem 7.

Corollary 9 : The sets of all contraction mappings of SS_n in matrix notation are symmetric ortho-stochastic, in other words it is reflection.

Proof: Suppose the permutation $\alpha = \{\alpha_i\}_{i=1}^{2} \in CSS_n$ and $\alpha_1 \neq \alpha_2$ by Lemma 2. Also, consider A to be matrix notation equivalent to α which is symmetric, then $A = A^T$ and $AA^T = A^T A$. Conversely, if $A \neq A^T \in SS_n$, obviously α is not a contraction mapping in SS_n. Thus, by Lemma 6 and Theorem 7 the result follows.

Corollary 10 : All symmetry elements of SS_n in matrix notation are ortho-stochastic but the converse is not true.
Proof: The result follows from Theorem 7.

Proposition 11: The sum and product of contraction elements in SS_n are symmetric.

Proof: Let $\alpha, \beta \in CSS_n$ and $\alpha \neq \beta$. Then, since $\alpha, \beta \in CSS_n$, we have that $\alpha = \alpha^T$ and $\beta = \beta^T$ by Corollary 10. Therefore, $\alpha\beta = \beta\alpha$ and $(\alpha\beta)^T = (\beta\alpha)^T$. Similarly, $(\alpha+\beta)^T = (\beta+\alpha)^T \Rightarrow \alpha+\beta = \beta+\alpha$. Hence, the result.

Discussion of Results

Firstly, the concept of trace, contraction and orthogonality were introduced on the elements of SS_n which leads to results in Proposition 1, Lemma 2 and Lemma 3 respectively. More so, Lemma 4 based on the concept of matrix representation relative to the properties of the basis of vectors from which Theorem 5 was deduced . Furthermore, the notion of ortho-stochastic stated in Lemma 6 were applied on SS_n and result in Theorem 7 was established. Lastly, the concept of reflection and contraction were also introduced and results on Corollary 9, 10 and were obtained.

Conclusion

This paper investigated some properties of inner product spaces on signed symmetric group in which results on ortho-stochactic and orthogonality were obtained. Furthermore, the concept of reflection and contraction were also introduced and some results were established.

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