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Stability Analysis of the Dynamics of Tuberculosis Model incorporating, Public Campaign, case Detection, Vaccine, Quarantine and Sanitarium as Control Strategies

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Abstract
In this paper, a tuberculosis (TB) model is formulated with the aim of determining the effects of public Campaign, Case detection, Vaccine, Quarantine and sanitarium on the spread and control of TB disease. The stability analysis shows and suggest that the control strategies
are locally asymptotically stable. The analysis shows that $R_0 < 1$ that means $R_{cvq} < 1$, $R_{vc} < 1$, implies $R_{cvq} < R_{vc} < R_v < 1$. Therefore, combination of case detection, vaccine, quarantine and sanitarium can reduce the endemicity of the infection rather than using a single strategy.

Keywords: Tuberculosis, transmission, basic reproduction number, management, quarantine.

Introduction

Tuberculosis (TB) is a contagious bacterial disease caused by inhaling the *Tubercle bacillus* in the droplet nucleus form. An infected person may have latent TB infections (noninfectious) or active TB infections (infectious). Only actively infected person who is sick with TB bacilli in lung is infectious. TB has been one of the greatest causes of human death throughout recorded. Mycobacterium tuberculosis infections can be dated to at least 1000 B.C. when people began congregating in urban environments (Castillo; 1997)

Tuberculosis (TB) is an airborne disease caused by mycobacterium tuberculosis (*Tuberccle bacillus).* Basically there are two types of TB: Pulmonary TB which affects the lung (*the commonest and infectious form of the disease)* and Extra-pulmonary TB that affects organs such as pleura, lymph nodes, spine, joints, abdomen or geneto-urnary track. TB is an infectious disease.

Its Transmission can occur when an infective persons contaminate the air by coughing, sneezing and spitting which generate a large number of small respiratory droplets that evaporate almost instantly into small droplet nuclei, disperse into the environment and implant themselves in the lung when inhaled. Although it can affect any part of the body, generally only active pulmonary and laryngeal TB pose a rest of transmission from one person to another. Tuberculosis is transmitted through

inhaling air containing droplet nuclei carrying the tubercle bacilli. In most cases, a competent immune system limits the multiplication of the *Tuberculosis bacilli*, although some bacilli remain dormant but viable, rendering a condition known as latent TB infection (LTBI), (CDC, 2000).

TB can be managed and control through the followings: Preventing infection and transmission through public campaign, vaccination, quarantine and sanitarium preventing the progression from latent infection to active TB through, drug treatment;

DOTS, routine – collected data, short course chemotherapy, early identification, isolation of infectious TB patients; effective engineering controls (environmental controls such as general ventilation, high – efficiency particulate air [HEPA) filters, or Ultraviolet Germicidal Irradiation [UVGGI]); the adoption of appropriate respiratory protection (surgical masks and particulate respirators such as HEPA Masks); health – care worker TB training, education, counseling and screening; and evaluation of the program's effectiveness. Pursuing high-quality DOTS

expansion and enhancement through: political commitment with increased and sustained financing, standardized treatment with supervision and patient support, effective drug supply and management system; addressing TB/HIV and other challenges through; implement of collaborative TB/HIV activities, address prisoners, refugees, other high-risk groups and special situations. Empowering TB patients and communities through advocacy, communication and social mobilization, community participation in TB care and patients charter for TB care.

Public campaign is an awareness to the right knowledge about TB on the part of both susceptible and infected persons as it relates to TB pathogenesis. This can be achieved or operate by addressing prisoners, refugees and other high risk groups and special situations, strengthening of health systems, engaging all care providers through public-public and public-private mix approaches. Also, through advocacy, communication and social mobilization

Slaa qanne (2010) Case detection play a great role in control of TB especially at latent stage. Case detection is the investigation of infectious disease through either laboratory means or Direct observation therapy strategy (DOTS). Conducted case detection over the 11 years between 1994 and 2005, a total of 26.5 million TB patients were diagnosed and reported under DOTS. That in 2005, DOTS programs worldwide reported 4.8 million new and relapsed cases, among which 2.3million were smear-positive. The smear-positive case detection was 60% (90% uncertainty limits, 52-69%) of the 3.9 million new cases estimate. the estimate of case detection was below the 70% target. And that the estimated case detection rate increased almost linearly from 11% globally in 1995 to 28%in 2000. They concluded that case detection has since accelerated from 2003 to 2004

Another strategy of control of TB is quarantine which refers to the separation and restriction of persons who, while not ill, have been exposed to an infectious agent and thereafter may become infectious. Quarantine may be used when a person has been exposed to a highly dangerous and infectious diseases. Quarantine can include a range of disease control strategies that may be used individually or in combination, include short-term voluntary home confinement; restrictions on travel by those who may have been exposed; and out of a geographic area. Quarantine also include other measures to control the spread of disease, such as restriction on the assembly of groups of people (e.g. school events); suspension of public gathering and closure of mass transit system or broad restrictions on travel by air, rail or water (CDC; 2014).

Slaa qanne(2010) Sanitarium is a medical facility for long term illness management. It is like a hospital where people who have had a serious illness like TB go so that health care workers can take care of them to recuperate. The efficiency of sanitarium treatment depends on early detection and reference of patients. Sanitarium can also be used as an educational center where TB patients receive both didactic and practical instructions

Materials and Methods

The total population is divided into seven compartments (*depending on the epidemiological status of individuals*) as follows: susceptible $S(t)$, Exposed $E(t)$, Infected $I(t)$, Quarantined $Q(t)$, Sanitarium $J(t)$, Vaccinated $V(t)$ and Recovered $R(t)$ compartments respectively. We refer to such model as SEIQJVR. Our parameters are presented and define as in the table below.

Table1: Definitions of Variables

Variables	Definitions
S(t)	The number of susceptible individuals at time, t
E(t)	The number of exposed individuals at time, t
I(t)	The number of infected individuals at time t
Q(t)	The number of individuals suspected with the symptoms of the disease at time t.
J(t)	The number of individuals in a sanitarium at time t
$V(t)$	The number of vaccinated individuals at time t.
$R(t)$	The number of recovered/treated individuals at time t

N(t) Total population at time, t Table 2: Definitions of parameters

Table 2: Definitions of parameters

Parameters	Definitions
ϕ_1	Rate of susceptible immigrants
μ	Natural mortality rate
$1/\mu$	The average life span of healthy individuals
d_1	TB induced death rate for undetected individuals
d_2	TB induced death rate for individuals in the sanitarium.
β	per capita transition rate from susceptible to exposed class.
k	The rate of progression of exposed individual to active TB
ω	Fraction of latently detected individuals quarantined
$1-\omega$	Fraction of the infected individuals who are not detected
r_2	Treatment rate for detected individuals in sanitarium
q	Fraction of detected and successfully treated in the Sanitarium
r_0	The rate of revert of recovered individuals to exposed class
r_3	The rate at which quarantined individuals are taken to Sanitarium
ϕ_2	The number of exposed latent immigrants
ϕ_3	The number of infected undetected immigrants
α	Transmission rate for the infected class
Λ	the number of new birth
ρ	the rate of suspected individuals with TB.
θ	the rate of expiration of vaccine efficacy
r_4	The rate at which the infected individuals go for treatment
β_1	The rate of movement from vaccinated compartment to susceptible compartment due to waning efficacy of vaccine.
r_1	The rate at which those successfully recovered go to susceptible class
β_2	The rate of vaccination of new born.
$(1-\beta_2)$	The proportion of new born that are not vaccinated

The following assumptions were made for the model:

(i) All patients including the immigrants suspected of TB case are quarantined.

(ii) Treatment of infected individuals occur in the sanitarium.

(iii) All infected individuals that are quarantined are taken to sanitarium for immediate treatment and enlightenment.

(iv) It is only infected individuals transmit the disease

(v) Screening is done to the exposed individual class.

(vi) Infected individuals are detected through either screening, other laboratory means and through sign and symptoms.

(vii) Those unsuccessfully treated remain infectious and die of the disease or naturally.

(viii) Those undetected may either die of the TB or natural mortality.

(ix) An immigrant can fall into susceptible, exposed, infected undetected respectively.

(x) There is screening of the immigrants at the border.

(xi) We also assume our recruitment rate to be constant

(xii) On recovery there is temporary immunity.

(xiii) Those successful recovered revert to the susceptible class.

(xiv) Those on vaccine become exposed on expiration of the vaccine efficacy.

Model Description and formulation

The total population is divided into seven compartments (*depending on the epidemiological status of individuals*) as follows: susceptible $S(t)$, Exposed $E(t)$, Infected $I(t)$, Quarantined $Q(t)$, Sanitarium $J(t)$, Vaccinated $V(t)$ and Recovered $R(t)$ compartments respectively. We refer to such model as SEIQJVR. The dynamics of the model is as follows; The recruitment into the vaccinated compartment is by birth at the rate β_2 and is reduced by $\beta_1 V$ due to the wanning efficacy of the vaccine and by natural death μV. The recruitment into susceptible compartments is by immigration, new born vaccinated at the birth and those not vaccinated, at the rates $\beta_1 V$, ϕ_1 and $(1-\beta_2)$ respectively. The susceptible population reduces by natural death and those that progresses to the exposed class at the rate μS and β IS respectively. The exposed compartment increases due to immigration at the rate ϕ_2 and those individuals that are on temporary immunity by the rate r_0R, while the exposed class reduces by the rate $(K+\alpha I)(1-\omega)$E and $(K+\alpha I)\,\omega$E due to transition to infected and sanitarium classes. The infected class increases by immigration at the rate ϕ_3 and reduces by (μ + d_1) I due to natural and disease induce deaths. The sanitarium class increases reduces by $(K+\alpha I)\,\omega$E and r_4I due to movement from exposed and infected classes and reduces by (μ + d_2) Q due to natural and disease induced deaths. The sanitarium compartment increases by $r_3 Q$ and reduces by q$r_2 J$ and (μ + d_2) J due to movement from quarantined into the sanitarium and movement out of sanitarium by treatment q$r_2 J$ and deaths (natural or disease induced) by (μ + d_2) J. The recovered class increases by $qr_2 J$ due to movement from sanitarium and reduces by $r_1 R$ and r_0R due to the movement to susceptible and exposed classes after recovery, and also reduces by μ R due to natural death.

General Transfer Diagram for the Modified Model

From our assumptions and the model description above we have the model flow diagram below

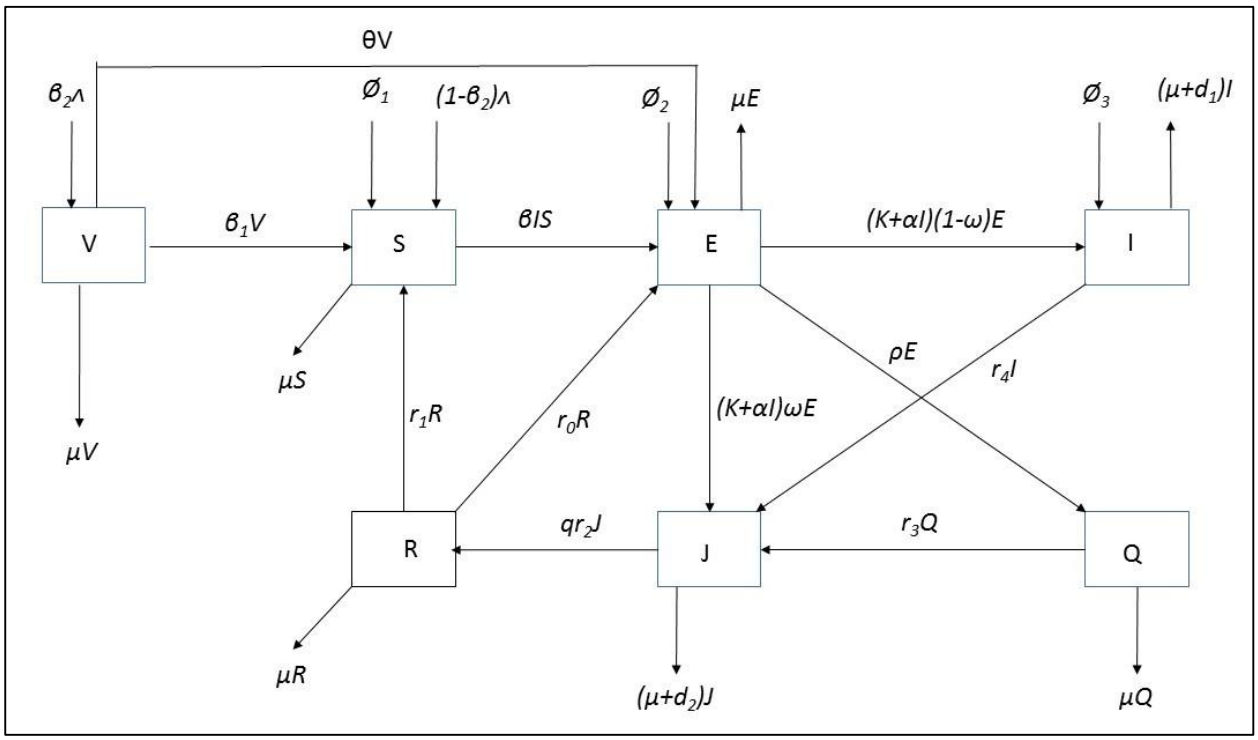

Figure 1: A general transfer diagram for the modified model (SEIQJVR

Model equations

Based on our assumptions and the flow diagram in figure 1, we develop the following set of ODEs.

$$\begin{aligned}
S'(t) &= (1-\beta_2)\wedge+\phi_1-\beta IS-\mu S+\beta_1 V+r_1 R\\
E'(t) &= \beta IS+\phi_2-(K+\alpha I)\omega E-(K+\alpha I)(1-\omega)E-\mu E+r_0 R-\rho E+\theta V\\
I'(t) &= (K+\alpha I)(1-\omega)E+\phi_3-(\mu+d_1)I-r_4 I\\
Q'(t) &= \rho E-r_3 Q-\mu Q\\
J'(t) &= (k+\alpha I)\omega E+r_3 Q+r_4 I-qr_2 J-(\mu+d_2)J\\
V'(t) &= \beta_2\wedge-\beta_1 V-\mu V-\theta V\\
R'(t) &= qr_2 J-\mu R-r_0 R-r_1 R\\
N(t) &= S(t)+E(t)+I(t)+Q(t)+J(t)+V(t)+R(t)
\end{aligned} \qquad (1)$$

Result and Discussion

Sensitivity Analysis

Disease-Free Equilibrium (DFE) point

The disease-free equilibrium is a steady state solution by which there is no disease or any intervention and a closed population.

Theorem: 1

A disease-free equilibrium state of the model (1) exist at the point

$$x_0 = (S,E,I,Q,J,V,R) = \left(\frac{(1-\beta_2)\Lambda+\phi_1}{\mu},0,0,0,0,\frac{\beta_2\Lambda}{\beta_1+\mu+\theta},0\right)$$

Proof

At equilibrium state the rate of change of variable is equal to zero

$$S'(t)=E'(t)=I'(t)=Q'(t)=J'(t)=V'(t)=R'(t)=0$$

This implies that $E=I=Q=J=V=R=0$

Therefore we have

$S'(t) = (1-\beta_2)\Lambda+\phi_1+\beta_1 V+r_1 R-\beta IS-\mu S$

Implies $(1-\beta_2)\Lambda+\phi_1+\beta_1 V+r_1 R-\beta IS-\mu S=0$

$(1-\beta_2)\Lambda+\phi_1-\mu S=0$

$\mu S=(1-\beta_2)\Lambda+\phi_1$

$S=\frac{(1-\beta_2)\Lambda+\phi_1}{\mu}$ (2)

The disease-free equilibrium will be used in computing the basic reproduction number. In the presence of vaccination, implies

$V'(t)=\beta_2\Lambda-\beta_1 V-\mu V-\theta V=0$

implies $V=\frac{\beta_2\Lambda}{\beta_1+\mu+\theta}$ (3)

In case of no disease $V\neq 0$, the total population is the sum of the susceptible and vaccinated individuals and is equal to

$\frac{(1-\beta_2)\Lambda+\phi_1}{\mu}+\frac{\beta_2\Lambda}{\beta_1+\mu+\theta}$ (4)

Therefore disease-free equilibrium (DFE) of model equation (1) is

$x_0=\left(\frac{(1-\beta_2)\Lambda+\phi_1}{\mu},0,0,0,0,\frac{\beta_2\Lambda}{\beta_1+\mu+\theta},0\right)$ (5)

Feasible Solution

Results show the feasible solution set which is positively invariant set of the model. The feasible solution set show that region is positively invariant and describes the region in which the solution of the TB model equations are biologically meaningful.

Suppose $X_0 = \{S,\ E,\ I,\ Q,\ J,\ V,\ R\} \in R_+^7 : N \leq \frac{(1-\beta_2)\Lambda+\phi_1}{\mu}$

$$N'(t) = (1-\beta_2)\Lambda + \varphi_1 + \varphi_2 - \mu S - \mu E + \varphi_3 - (\mu + d_1)I + \mu Q - (\mu - d_2)J - \mu V - \mu R = 0$$

$$(1-\beta_2)\Lambda + \phi_1 + \phi_2 + \phi_3 - \mu S - \mu E - \mu R - \mu V - \mu Q - (\mu - d_2)J - (\mu + d_1)I = 0$$

$$N'(t) = (1-\beta_2)\Lambda + \phi_1 + \phi_2 + \phi_3 - \mu(S + E + R + V + I) - [(\mu - d_2)J + (\mu + d_1)I] - \mu Q = 0$$

In the absence of immigration, TB induced death and vaccination implies $\phi_1 = \phi_2 = \phi_3 = 0$ and $d_1 = d_2 = 0$ and $V = 0$ implies

$$N'(t) = (1-\beta_2)\Lambda - \mu(S + E + I + Q + J + V + R)$$

$$\Rightarrow \frac{dN}{dt} = (1-\beta_2)\Lambda - \mu N \tag{6}$$

Applying the Birhoft and Rotas (1989) theorem on differentiantial inequality (6)
By separation of variable

$$\frac{dN}{(1-\beta_2)\Lambda - \mu N} \leq dt$$

Integrating bot side we have

$$\int \frac{dN}{(1-B_2)\Lambda - \mu N} \geq \int dt = e^{-\mu t} + \frac{(1-B_2)\Lambda}{\mu}(1 - e^{-\mu t})$$

i.e. $N(t) \leq N(0)e^{-\mu t} + \frac{(1-B_2)\Lambda}{\mu}(1 - e^{-\mu t})$

If $N(0) \leq \frac{(1-B_2)\Lambda}{\mu}$ then $N(t) \leq \frac{(1-B_2)\Lambda+\phi_1}{\mu}$ (7)

Thus X_0 is a positively invariant set under the system in (1). Hence no solution pathleave through any boundary of X_0, therefore solutions remain non-negative for non-negative initial conditions. Therefore the model equation (1) is mathematically and epidemiologically well posed.

Positivity of Solution

Theorem: 2

Let the initial data be $(S, E, I, Q, J, V, R)(0) \geq 0 \in x_1$. Then, the solution set of equation (1), is positive for all $t > 0$.

From the first equation of (1), we have

$$\frac{ds}{dt} = (1-\beta_2)\Lambda + \phi_1 + \beta_1 V + r_1 R - S(\beta I + \mu)$$

$$\geq -(\beta I + \mu)S \tag{8}$$

Integrating equation (8), we have

$$\int \frac{dS}{S} \geq -\int (\beta I + \mu)dt$$

$$LnS \geq -\int (\beta I + \mu)dt,\ S = e^{-\mu t - \beta \int I dt}$$

$S(t) \geq S(0)e^{-\mu t} - \beta\int Idt \geq 0$, since $e^{-\mu t-\beta\int Idt}$

From the second equation of (1), we have

$$\frac{dE}{dt} = \beta IS + \phi_2 + r_0 R - (K+\alpha I)\omega E - \mu E - \rho E + \theta V \geq -[(\mu+\rho)+(k+\alpha I)\omega + (K+\mu I)(1-\omega)]E \quad (9)$$

Integrating equation (9)

$$E(t) \geq e^{-(\mu+\rho)t} + e^{-K\omega t-\int \alpha I dt} + e^{-K(1-\omega)-\alpha\omega\int Idt} + e^{-[\mu+\rho+k\omega+k+k\omega]t-(2\alpha+\alpha\omega)\int Idt}$$

$$E(t) \geq E(0), e^{-(\mu+\rho+k)t-\alpha(2+\omega)\int Idt}, \text{ since } (\mu+\rho+k)t - \alpha(2+\omega)\int Idt > 0$$

From the third equation of (1), we have

$$\frac{dI}{dt} = (k+\alpha I)(1-w)E + \phi_3 - (\mu+d_1)I - r_4 I$$

$$\geq -(\mu+d_1+r_4)I \quad (10)$$

Integrating equation (10), we have

$$I(t) \geq I(0)e^{-(\mu+d_1+r_4)t} \geq 0, \text{Since } \mu + d_1 + r_4 > 0$$

From the fourth equation of (1), we have

$$\frac{dQ}{dt} = \rho E - r_3 Q - \mu Q \geq -(r_3+\mu)Q \quad (11)$$

When we integrating equation (11), we get

$$Q(t) \geq Q(0)e^{-(r_3+\mu)t} \geq 0, \text{ Since } r_3 + \mu > 0$$

From the fifth equation of (1), we have

$$\frac{dJ}{dt} = (k+\alpha I)\omega E + r_3 Q + r_4 I - qr_2 J - (\mu+d_2)J$$

$$\geq -(qr_2+\mu+d_2) \quad (12)$$

Integrating equation (12), we have

$J(t) \geq J(0)e^{-(qr_2+\mu+d_2)t} \geq 0$, since $qr_2 + \mu + d_2 > 0$

From the sixth equation of (1), we have

$$\frac{dv}{dt} = \beta_2\Lambda - \beta_1 v - \theta v - \mu v \geq -(\beta_1+\theta+\mu)v \quad (13)$$

Integrating equation (13), we have

$V(t) \geq V(0)e^{-(\beta_1+\theta+\mu)} \geq 0$ Since $(\beta_1+\theta+\mu) > 0$

From the last equation of (1), we have $\frac{dR}{dt} = qr_2 J - r_0 R - r_1 R - \mu R \geq -(r_0+r_1+\mu)R$ (14)

Integrating (14) we have $R(t) \geq R(0)e^{-(r_0+r_1+\mu)}$, since $(r_0+r_1+\mu) > 0$

Thus, we have shown that all variables are positive for all time $t > 0$.

Basic Reproductive number

The global behavior of the proposed model (1) crucially depends on the basic reproduction number. Diekman et al., (1990), the effective reproductive number is the secondary infection caused by a single infective TB individual introduced into a population of susceptible individuals. We calculate the basic reproduction number by using the next generation operator methods on the system (14).

f_i is the transmission matrix while v_i is the transition matrix.

Let $S = \frac{(1-\beta_2)\Lambda+\phi_1}{\mu}$

We linearized equation (1)

$$\left.\begin{aligned}
S'(t) &= (1-\beta_2)\Lambda + \phi_1 + \beta_1 V + r_1 R - \tfrac{\beta I(1-\beta_2)\Lambda+\phi_1}{\mu} - (1-\beta_2)\Lambda + \phi_1 \\
E'(t) &= \tfrac{\beta I(1-\beta_2)\Lambda+\phi_1}{\mu} + \phi_2 + r_0 R - (k+\alpha I)\omega E - (k+\alpha I)(1-\omega)E - \mu E + \rho E + \theta v \\
I'(t) &= (k+\alpha I)(1-\omega)E + \varphi_3 - (\mu + d_1)I - r_4 I \\
Q'(t) &= \rho E - r_3 Q - \mu Q \\
J'(t) &= (k+\alpha I)\omega E + r_3 Q + r_4 I - q r_2 I - (\mu + d_2)J \\
V'(t) &= \beta_2 \Lambda - \beta_1 v - \theta v - \mu v \\
R'(t) &= q r_2 J - r_0 R - r_1 R - \mu R
\end{aligned}\right\} \quad (15)$$

From equation (3.14) we obtain the expression of F and V at the disease free equilibrium point x_0 O.Diekmann etal (2010)

$$f_i = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} \frac{\beta I(1-\beta_2)\Lambda + \phi_1}{\mu} \\ (k+\alpha I)(1-\omega)E \\ \rho E \\ (k+\alpha I)\omega E \\ \beta_2 \Lambda \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(x_0)}{\partial E} & \frac{\partial f_1}{\partial I} & \frac{\partial f_1}{\partial Q} & \frac{\partial f_1}{\partial J} & \frac{\partial f_1}{\partial V} \\ \frac{\partial f_2(x_0)}{\partial E} & \frac{\partial f_2}{\partial I} & \frac{\partial f_2}{\partial Q} & \frac{\partial f_2}{\partial J} & \frac{\partial f_2}{\partial V} \\ \frac{\partial f_3(x_0)}{\partial E} & \frac{\partial f_3}{\partial I} & \frac{\partial f_3}{\partial Q} & \frac{\partial f_3}{\partial J} & \frac{\partial f_3}{\partial V} \\ \frac{\partial f_4(x_0)}{\partial E} & \frac{\partial f_4}{\partial I} & \frac{\partial f_4}{\partial Q} & \frac{\partial f_4}{\partial J} & \frac{\partial f_4}{\partial V} \\ \frac{\partial f_5(x_0)}{\partial E} & \frac{\partial f_5}{\partial I} & \frac{\partial f_5}{\partial Q} & \frac{\partial f_5}{\partial J} & \frac{\partial f_5}{\partial V} \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} -E(\mu+\rho) \\ -I(\mu + d_1 - r_4) \\ \rho E - r_3 Q - \mu Q \\ r_3 Q + r_4 I - q r_2 J - (\mu + d_2)J \\ -\beta_1 v - \theta v - \mu v \end{bmatrix} = \begin{bmatrix} \frac{\partial v_1(x_0)}{\partial E} & \frac{\partial v_1}{\partial I} & \frac{\partial v_1}{\partial Q} & \frac{\partial v_1}{\partial J} & \frac{\partial v_1}{\partial V} \\ \frac{\partial v_2(x_0)}{\partial E} & \frac{\partial v_2}{\partial I} & \frac{\partial v_2}{\partial Q} & \frac{\partial v_2}{\partial J} & \frac{\partial v_2}{\partial V} \\ \frac{\partial v_3(x_0)}{\partial E} & \frac{\partial v_3}{\partial I} & \frac{\partial v_3}{\partial Q} & \frac{\partial v_3}{\partial J} & \frac{\partial v_3}{\partial V} \\ \frac{\partial v_4(x_0)}{\partial E} & \frac{\partial v_4}{\partial I} & \frac{\partial v_4}{\partial Q} & \frac{\partial v_4}{\partial J} & \frac{\partial v_4}{\partial V} \\ \frac{\partial v_5(x_0)}{\partial E} & \frac{\partial v_5}{\partial I} & \frac{\partial v_5}{\partial Q} & \frac{\partial v_5}{\partial J} & \frac{\partial v_5}{\partial V} \end{bmatrix}$$

$$F = \frac{\partial f_i(x_0)}{\partial x_j} = \begin{bmatrix} 0 & \beta(1-\beta_2)\Lambda & 0 & 0 & 0 \\ k - k\omega & 0 & 0 & 0 & 0 \\ \rho & 0 & 0 & 0 & 0 \\ k\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We find our G which is $G = FV^{-1}$.

$$\begin{bmatrix} 0 & \frac{\beta(1-\beta_2)\Lambda}{\mu} & 0 & 0 & 0 \\ k(1-\omega) & 0 & 0 & 0 & 0 \\ \rho & 0 & 0 & 0 & 0 \\ km & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} -\frac{1}{\mu+\rho}, 0, 0, \\ 0, 0 \end{bmatrix}, \\ \begin{bmatrix} 0, -\frac{1}{\mu+d_1-r_4}, 0, 0, 0 \end{bmatrix}, \\ \begin{bmatrix} -\frac{\rho}{(r_3+\mu)(\mu+\rho)}, 0, -\frac{1}{r_3+\mu}, 0, 0 \end{bmatrix}, \\ \begin{bmatrix} -\frac{r_3\rho}{(r_3+\mu)(\mu+\rho)(\mu+d_2)}, \\ -\frac{r_4}{(\mu+d_1-r_4)(\mu+d_2)}, \\ -\frac{r_3}{(r_3+\mu)(\mu+d_2)}, -\frac{1}{\mu+d_2}, 0 \end{bmatrix}, \\ \begin{bmatrix} 0, 0, 0, 0, -\frac{1}{\beta_1+\theta+\mu} \end{bmatrix} \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} -\frac{1}{\mu+\rho} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\mu+d_1-r_4} & 0 & 0 & 0 \\ -\frac{\rho}{(r_3+\mu)(\mu+\rho)} & 0 & \frac{1}{r_3+\mu} & 0 & 0 \\ -\frac{r_3\rho}{(r_3+\mu)(\mu+\rho)(\mu+d_2)} & -\frac{r_4}{(\mu+d_1-r_4)(\mu+d_2)} & -\frac{r_3}{(r_3+\mu)(\mu+d_2)} & \frac{-1}{\mu+d_2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\beta_1+\theta+\mu} \end{bmatrix} \text{W}$$

$$G = FV^{-1} = \begin{bmatrix} 0 & -\frac{\beta(1-\beta_2)\Lambda}{\mu(\mu+d_1-r_4)} & 0 & 0 & 0 \\ -\frac{k(1-\omega)}{\mu+\rho} & 0 & 0 & 0 & 0 \\ -\frac{\rho}{\mu+\rho} & 0 & 0 & 0 & 0 \\ -\frac{k\omega}{\mu+\rho} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We find the matrix of $G - I\lambda$ where I is the identity matrix.

$$\begin{bmatrix} 0 & -\frac{\beta\left(1-\beta_2\right)\Lambda}{\mu\left(\mu+d_1-r_4\right)} & 0 & 0 & 0 \\ -\frac{k(1-\omega)}{\mu+\rho} & 0 & 0 & 0 & 0 \\ -\frac{\rho}{\mu+\rho} & 0 & 0 & 0 & 0 \\ -\frac{km}{\mu+\rho} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\lambda\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G - I\lambda = \begin{bmatrix} -\lambda & -\frac{\beta(1-\beta_2)\Lambda}{\mu(\mu+d_1-r_4)} & 0 & 0 & 0 \\ -\frac{k(1-\omega)}{\mu+\rho} & -\lambda & 0 & 0 & 0 \\ -\frac{\rho}{\mu+\rho} & 0 & -\lambda & 0 & 0 \\ -\frac{k\omega}{\mu+\rho} & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & -\lambda \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & -\frac{\beta\left(1-\beta_2\right)\Lambda}{\mu\left(\mu+d_1-r_4\right)} & 0 & 0 & 0 \\ -\frac{k(1-\omega)}{\mu+\rho} & -\lambda & 0 & 0 & 0 \\ -\frac{\rho}{\mu+\rho} & 0 & -\lambda & 0 & 0 \\ -\frac{km}{\mu+\rho} & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & -\lambda \end{bmatrix} \overset{\text{simplify}}{=}$$

We find the determinant of $G - I\lambda$ matrix to obtain our characteristics polynomial.

$$\begin{bmatrix} -\lambda & -\frac{\beta(1-\beta_2)\Lambda}{\mu(\mu+d_1-r_4)} & 0 & 0 & 0 \\ -\frac{k(1-\omega)}{\mu+\rho} & -\lambda & 0 & 0 & 0 \\ -\frac{\rho}{\mu+\rho} & 0 & -\lambda & 0 & 0 \\ -\frac{km}{\mu+\rho} & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & -\lambda \end{bmatrix} \xrightarrow{\text{characteristic polynomial}} 6\lambda^5$$

$$-\frac{1}{(\mu+\rho)\mu(\mu+d_1-r_4)}\left(\left(-10\lambda^2\mu^3-10\lambda^2\mu^2 d_1+10\lambda^2\mu^2 r_4-10\lambda^2\mu^2\rho-10\lambda^2\mu\rho d_1\right.\right.$$
$$\left.+10\lambda^2\mu\rho r_4+k(1-\omega)\beta(1-\beta_2)\Lambda\right)\lambda^3\Big)-\frac{1}{(\mu+\rho)\mu(\mu+d_1-r_4)}\left(\lambda^3\left(-10\lambda^2\mu^3\right.\right.$$
$$\left.\left.-10\lambda^2\mu^2 d_1+10\lambda^2\mu^2 r_4-10\lambda^2\mu^2\rho-10\lambda^2\mu\rho d_1+10\lambda^2\mu\rho r_4+3k(1-\omega)\beta(1-\beta_2)\Lambda\right)\right.$$
$$\Big)-\frac{1}{(\mu+\rho)\mu(\mu+d_1-r_4)}\left(\lambda^3\left(-5\lambda^2\mu^3-5\lambda^2\mu^2 d_1+5\lambda^2\mu^2 r_4-5\lambda^2\mu^2\rho-5\lambda^2\mu\rho d_1\right.\right.$$
$$\left.\left.+5\lambda^2\mu\rho r_4+3k(1-\omega)\beta(1-\beta_2)\Lambda\right)\right)$$
$$-\frac{\lambda^3\left(k(1-\omega)\beta(1-\beta_2)\Lambda-\lambda^2\mu^3-\lambda^2\mu^2 d_1+\lambda^2\mu^2 r_4-\lambda^2\mu^2\rho-\lambda^2\mu\rho d_1+\lambda^2\mu\rho r_4\right)}{(\mu+\rho)\mu(\mu+d_1-r_4)}$$

$$\xrightarrow{\text{solve for lambda}}\left[[\lambda=0],[\lambda=0],[\lambda=0],\left[\lambda=\frac{1}{2}\frac{\sqrt{\mu k(1-\omega)\beta(1-\beta_2)\Lambda\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}}{\mu\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}\right],\left[\lambda=-\frac{1}{2}\frac{\sqrt{\mu k(1-\omega)\beta(1-\beta_2)\Lambda\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}}{\mu\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}\right]\right]$$

The eigen values (roots of the equation obtained) are

$$\lambda_1=0,\ \lambda_2=0,\ \lambda_3=0,$$

$$\lambda_4=\frac{1}{2}\frac{\sqrt{\mu k(1-\omega)\beta(1-\beta_2)\Lambda\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}}{\mu\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}$$

$$\lambda_5=-\frac{1}{2}\frac{\sqrt{\mu k(1-\omega)\beta(1-\beta_2)\Lambda\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}}{\mu\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}$$

The eigenvalues λ of the matrix $G-I\lambda$ is computed from the characteristics equation $|G-I\lambda|$. We obtain λi from $i=1,2,\cdots,5$ as $\lambda_1=\lambda_2=\lambda_3=0$ and

$$\lambda_4=\frac{1}{2}\frac{\sqrt{\mu k(1-\omega)\beta(1-\beta_2)\Lambda\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}}{\mu\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}$$

$$\lambda_5=-\frac{1}{2}\frac{\sqrt{\mu k(1-\omega)\beta(1-\beta_2)\Lambda\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}}{\mu\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}$$

From the value of λ above it is clearly that λ_4 is the dominant (largest) eigenvalue. Therefore λ_4 is the effective reproduction number (R_0) of our model system (1) with vaccine, case defection, quarantine, sanitarium as control strategies.

Therefore:

$$R_0=\frac{1}{2}\frac{\sqrt{\mu k(1-\omega)\beta(1-\beta_2)\Lambda\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r\right)}}{\mu\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}$$

In the absence of treatment $(i.e\ r_2=r_4)$ the effective reproductive number wit case detection, vaccine and quarantined only is given by

$$R_{cvq}=\frac{1}{2}\frac{\sqrt{\mu k(1-\omega)\beta(1-\beta_2)\Lambda\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_{14}\right)}}{\mu\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}$$

In the absence of quarantined $(i.e\ r_3=p)$ the effective reproduction number with case defection and vaccine only is given by

$$R_{vc}=\frac{1}{2}\frac{\sqrt{\mu(1-\omega)\beta(1-\beta_2)\Lambda\left(\mu^2+\mu d_1\right)}}{\mu\left(\mu^2+\mu d_1\right)}$$

In the absence of case defection

i.e. $\omega=0$

$$R_v=\frac{1}{2}\frac{\sqrt{\mu\beta(1-\beta_2)\Lambda\left(\mu^2+\mu d_1\right)}}{\mu\left(\mu^2+\mu d_1\right)}$$

We also note that $\mu\left[\beta(1-\beta_2)\Lambda\left(\mu^2+\mu d_1\right)\right]$ is multiplied by term $(1-\omega)$ which means $R_{cv}<R_v$ therefore we conclude that the endemicity of the infection or the intensity of the infection is reduced more when we combined case detection and vaccine rather than vaccine only as control strategy. Finally, if there is no vaccine i.e $\beta_2=\rho=\omega=r_4=0$ we have

$$R_v=\frac{1}{2}\frac{\sqrt{\mu\beta\Lambda\left(\mu^2+\mu d_1\right)}}{\mu\left(\mu^2+\mu d_1\right)}$$

But we observe that the term $\Lambda\left(\mu^2+\mu d_1\right)$ is multiplied by the proportion $(1-\beta_2)$ which means $R_{cvq}<R_v$ Therefore we can conclude that introduction of the case detection vaccine and quarantined can reduce the endemicity of the infection rather than using vaccine only. Therefore our reproductive number (R_0) of our model system (1) is λ_4

$$R_0 = \frac{1}{2}\frac{\sqrt{\mu k(1-\omega)\beta(1-\beta_2)\Lambda\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}}{\mu\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}$$

With the combination of case defection vaccine, quarantined and sanitarium as control strategies.

***Local Stability of the DFE*(x_0)**

Theorem: 3

For the disease free equilibrium point of the model system (3.2) to be

i. Locally asymptotically stable if $R_0 < 1$

ii. Locally asymptotically unstable if $R_0 > 1$

If and only if $J(x_0)$ has negative trace and positive determinant.

Trace of a square matrix$A = [Ai_j]Ef^{n,n}$, denoted as$tr(A)E\ F$, is the sum of its diagonal elements, that is, the scalar given by $Ar(A) = A_{11} + - - - + A_{nn}$ (Nyerere; 2014)

Proof

Here we take the partial differentiation of (3.2) with respect to (S, E, I, Q, J, V, R) at the disease-free equilibrium which given us:

$$\begin{bmatrix} -\mu & 0 & \frac{\beta\Lambda(1-\beta_2)}{\mu} & 0 & 0 & \beta_1 & r_1 \\ \beta I & -a & \frac{\beta\Lambda(1-\beta_2)}{\mu} & 0 & 0 & \theta & r_0 \\ 0 & k(1-\omega) & -b & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & -(\mu+r_3) & 0 & 0 & 0 \\ 0 & k\omega & r_4 & r_3 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -d & 0 \\ 0 & 0 & 0 & 0 & qr_2 & 0 & -e \end{bmatrix}$$

Where $a = (k+\mu+\rho), b = (\mu+d_1+r_4), c = (qr_2+\mu+d_2), d = (\beta_1+\mu+\theta)$ and $e = (r_1+r_0+\mu)$

$g = -(\mu + r_3)$

The trace of our matrix $J(x_0)$ is given by

$$T_r\ J(x_0) = -\mu-(a+b+c+d+e)-(\mu+r_3)$$

$$T_r\ J(x_0) = -\mu-(k+\mu+\rho)-(\mu+d_1+r_4)-(qr_2+\mu+d_2)$$
$$-(\beta_1+\mu+\theta)-(r_1+r_0+\mu)-(\mu-r_3) < 0$$

Implies $T_r\ J(x_0) < 0$

Now we find the determinant of $J(x_0)$ i. e.

$$|J(x_0)| = \begin{vmatrix} -\mu & 0 & \frac{\beta\Lambda(1-\beta_2)}{\mu} & 0 & 0 & \beta_1 & r_2 \\ \beta I & -a & \frac{\beta\Lambda(1-\beta_2)}{\mu} & 0 & 0 & 0 & r_0 \\ 0 & k(1=\omega) & -b & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & -(\mu+r_3) & 0 & 0 & 0 \\ 0 & k\omega & r_4 & r_3 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -d & 0 \\ 0 & 0 & 0 & 0 & qr_2 & 0 & -e \end{vmatrix} > 0$$

Therefore $|J(x_0)| > 0$

This implies that the determinant of our partial differentiation matrix, is positive and the trace of our matrix $J(x_0)$ is less than zero implies $R_0 < 1$ hence the model system (1) is locally asymptotically stable at the disease free equilibrium x_0. Going by the stated theorem *TB* can be eliminated from the environment wen $R_0 < 1$ or it implies on average, an infected individual produce less than one new infected individual during the course of the disease and that it cannot grow.

Endemic equilibrium point (EEP)

It is obtained by setting the right hand side of each equation (1) equal to zero. In the presence of infection, that is, $I \neq 0$ and $E \neq 0$, the model has a non-trivial equilibrium point given by

$$E_0 = \left(S^*, E^*, I^*, Q^*, J^*, V^*, R^*\right) \neq 0$$

From the equation of infected class of (1)

$$I'(t) = (k + \alpha I)(1-\omega)E + \phi_3 - (\mu + d_1)I - r_4 I = 0$$

Suppose, $I = 1$, we have

$$(k + \alpha I)(1-\omega)E + \phi_3 - (\mu + d_1) - r_4 I = 0$$

$$(k + \alpha)(1-\omega)E = \mu + d_1 + r_4 - \phi_3$$

Making E the subject of the expression:

$$E^* = \frac{\mu + d_1 + r_4 - \phi_3}{(k + \alpha)(1-\omega)} \tag{16}$$

From the same expression

$$(k + \alpha I)(1-\omega)E^* + \phi_3 - (\mu + d_1)I - r_4 I = 0$$

$$(k - \omega k + \alpha I - \alpha\omega I)E^* + \phi_3 - (\mu + d_1 + r_4)I = 0$$

$$I^* = (k - \omega k + \alpha I - \alpha\omega I)E^* + \phi_3$$

Substituting E^*, we have

$$(k - \omega k + \alpha I - \alpha\omega I)\left(\frac{\mu + d_1 + r_4 - \phi_3}{(k + \alpha)(1-\omega)}\right) + \phi_3 - (\mu + d_1 + r_4)I = 0$$

$$\phi_3 + k(1-\omega) = I\left[\frac{\mu + d_1 + r_4 - \phi_3}{(k + \alpha)(1-\omega)}\right] + (\mu + d_1 + r_4 - \alpha) = 0$$

$$= \frac{\phi_3 + k(1-\omega)}{\alpha\omega}\left[\frac{(k + \alpha)(1-\omega)}{\mu + d_1 + r_4 - \phi_3} + \left(\frac{\phi_3 + k(1-\omega)}{\mu + d_1 + r_4 - \alpha}\right)\right]$$

$$I^* = \frac{\phi_3 + k(1-\omega)(k + \alpha)(1-\omega)}{\alpha\omega(\mu + d_1 + r_4 - \phi_3)} + \left(\frac{(\phi_3 + k(1-\omega))^2}{\alpha\omega(\mu + d_1 + r_4 - \alpha)}\right) \tag{17}$$

Also considering $Q'(t) = \rho E - r_3 Q - \mu Q$, we have

$$\rho E - r_3 Q - \mu Q = 0$$

$$\rho E = Q(r_3 + \mu)$$

$$Q^* = \frac{\rho E^*}{(r_3 + \mu)} \tag{18}$$

Substituting E^*, we have

$$Q^* = \frac{\rho(\mu + d_1 + r_4 - \phi_3)}{(k + \alpha)(1 - \omega)(r_3 + \mu)}$$

From $J'(t) = (k + \alpha I)\omega E + r_3 Q + r_4 I - q r_2 J - (\mu + d_2)J = 0$

We have

$$(k + \alpha I)\omega E + r_3 Q + r_4 I - q r_2 J - (\mu + d_2)J = 0$$

Let $E^* = x$ and $I^* = z$

Substituting E^* and I^*, we have

$$(k + \alpha z)\omega x + r_3 Q^* + r_4 z = (q r_2 + \mu + d_2)J$$

Solving for J, we have

$$J^* = \frac{(k + \alpha z)(\omega x + r_3 Q^* + r_4 z)}{(q r_2 + \mu + d_2)} \tag{19}$$

Similarly from

$$q r_2 J - r_0 R - r_1 R - \mu R = 0$$

$$q r_2 J^* = R(r_0 + r_1 + \mu) = 0$$

Solving for R, we have

$$R^* = \frac{q r_2 J^*}{(r_0 + r_1 + \mu)} \tag{20}$$

Considering the equation

$$\beta_2 \Lambda - \beta_1 V - \theta V - \mu V = 0$$

$$\beta_2 \Lambda = V(\beta_1 - \theta - \mu) = 0$$

$$V^* = \frac{\beta_2 \Lambda}{(\beta_1 - \theta_2 - \mu)} \tag{21}$$

Finally from the first equation of the model (1)

$$(1 - \beta_2)\Lambda + \phi_1 + \beta_1 V + r_1 R - \beta I S - \mu S = 0$$

We have

$$(1 - \beta_2)\Lambda + \phi_1 + \beta_1 V^* + r_1 R^* = S(\beta I + \mu)$$

$$S^* = \frac{(1 - \beta_2)\Lambda + \phi_1 + \beta_1 V^* + r_1 R^*}{(\beta z + \mu)} \tag{22}$$

The endemic equilibrium point E_0 of our system (1) are equations (22, 16, 17, 19, 21, 20, 18,) respectively.

Conclusion

In this work, a deterministic TB model for transmission of TB is formulated to access the effects of case detection, vaccine, quarantine and sanitarium. It has been proved that the feasible solution region is positively invariant, which makes the model equation biologically meaningful also it

has been shown that all variables are positive for all time t >0.

We obtain our basic reproduction number

$$R_0 = \frac{1}{2}\frac{\sqrt{\mu k(1-\omega)\beta(1-\beta_2)\Lambda\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r\right)}}{\mu\left(\mu^2+\mu d_1-\mu r_4+\rho\mu+\rho d_1-\rho r_4\right)}.$$

The analysis of our basic reproduction number shows that $R_{cvq} < R_{vc} < R_v$ implies that combination of case detection, vaccine, quarantine and sanitarium as control strategies in the transmission of tuberculosis reduces the endemicity of the infection and suggest that the implementation of the strategies so a good step for the eradication for tuberculosis. We have shown that our model equations locally asymptotically stable at the disease -free equilibrium. We also show that our model is locally stable since $T_r\ J(x_0) < 0$ and $|J(x_0)| > 0$.

Recommendation

Our recommendation to government and health workers agencies is to address awareness program on early detection, intensify screening program at the borders, completion of treatment by the patients and creation of TB screening units at various strategic check point of both immigration and custom Department

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