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Analysis of States, Controls and Cost Functional of Two Dimensional Energized Wave Equation Up to Ten Nodal Points

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ABSTRACT

The wave equation is an important second order linear partial differential equation for description of waves.This research obtains the solution of the control, state and the cost functional in two dimensional energized wave equation up to ten nodal points. The fourier solution proposed by Duchateau and Zachmann (1986) for deriving the general equations was applied to the problems of two dimensional quadratic functional Min $J(z,u) = \int_0^1\int_0^1\int_0^1[Z^2(x,y,t) + U^2(x,y,t)]dxdydt$ Subject to $\frac{\partial^2 z(x,y,t)}{\partial t} + \frac{\partial z(x,y,t)}{\partial t} = \frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2} + u(x,y,t)$ The finite element techniquewas then used on the resulting system to obtain the states, controls and the cost functional at different levels of discretization up to ten nodal points. The findings in the one dimensional case hold.

KEYWORDS: PartialDifferential Equation, Wave Equation, Nodal Point, Optimal Control, Optimal State

Introduction

The research work is an extension of the work of Bawa (2003), applicable to the two dimensional optimal control of wave equation with energy effect incorporated with the optimization of a quadratic functional

Min $J(z,u) = \int_0^1\int_0^1\int_0^1[Z^2(x,y,t) + U^2(x,y,t)]dxdydt$.

Subject to

$$\frac{\partial^2 z(x,y,t)}{\partial t} + \frac{\partial z(x,y,t)}{\partial t} = \frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2} + u(x,y,t)$$

The applications of optimization methods to equations in mathematical physics have been considered byReju et al (2001). They applied the extended conjugate gradient method to the control problems of diffusion, fluid dynamics and wave propagation. Other works include Reju (1995), and Waziri (2004). In this work, the finite element technique is used to find the optimal states, controls, and the cost functional of wave equation with energy effects in two dimensional case up to ten nodal points following the transformational steps in Binder (1911), Pain (1997), Raisinghania (2010) and Schmidt (1924).

According to Rao (1985), the finite element technique can be concisely defined as an approximation method of solving complex problems where the solution region is taken as an assemblage of many small – interconnected sub–regions called finite elements.

Wave Equation With Energy Effects in Two Dimension

Using the procedural steps in Bawa (2003), the two – dimensionaloptimal control problem is considered.The two dimensional wave equations with energy effect is given as

$$\left(\frac{1}{c^2}\right)\frac{\partial^2 z(x,y,t)}{\partial t^2}+\left(\frac{1}{d}\right)\frac{\partial z(x,y,t)}{\partial t}=\frac{\partial^2 z(x,y,t)}{\partial x^2}+\frac{\partial^2 z(x,y,t)}{\partial y^2} \qquad (21)$$

The optimization problem under consideration is given by

$$\text{Min } \mathrm{J}[Z,U]=\int_0^1\int_0^1\int_0^1[\mathrm{Z}^2(x,y,t)+u^2(x,y,t)]\mathrm{dxdydt} \qquad (2.2)$$

Subject to

$$\frac{\partial^2 z(x,y,t)}{\partial t}+\frac{\partial z(x,y,t)}{\partial t}=\frac{\partial^2 z(x,y,t)}{\partial x^2}+\frac{\partial^2 z(x,y,t)}{\partial y^2}+u(x,y,t)$$

With boundary and initial conditions

$$Z(0,y,t)=Z(1,y,t),\qquad 0\le x\le 1 \qquad (2.3)$$
$$Z(x,0,t)=Z(x,1,t),\qquad 0\le y\le 1$$
$$Z(x,y,0)=Z(x,y,1),\qquad 0\le t\le 1$$

Following Singh and Titli(198),the Hamiltonian for (2.2) and (2.3) is given as

$$\mathrm{H}=Z^2(x,y,t)+U^2(x,y,t)+$$
$$\lambda^{\mathrm{T}}\left[Z_{XX}(x,y,t)+Z_{yy}(x,y,t)+U(x,y,t)\right] \qquad (2.4)$$

Where $\lambda^{\mathrm{T}}=\lambda^{\mathrm{T}}(\mathrm{t})$

Setting

$$\mathrm{f}\,(z,u)=Z_{XX}(x,y,t)+Z_{yy}(x,y,t)+U(x,y,t)$$
$$\mathrm{g}(z,u)=Z^2(x,y,t)+U^2(x,y,t)$$

The first order necessary conditions for optimality is

$$Z_t(x,y,t)=\mathrm{H}_\lambda(x,y,t)$$
$$=Z_{XX}(x,y,t)+Z_{yy}(x,y,t)+U(x,y,t)$$
$$=\mathrm{f}\big(Z(x,y,t),U(x,y,t)\big) \qquad (2.5)$$
$$\lambda_t=-\mathrm{H}_z=[\mathrm{f}_z]^{\mathrm{T}}\lambda-\mathrm{g}_z=-2z(x,y,t) \qquad (2.6)$$

$$\mathrm{H_u}=0$$
$$\text{or}[\mathrm{f_u}]\quad \lambda^{\mathrm{T}}+\mathrm{g_u}=0 \qquad (2.7)$$

Where H = $\mathrm{g}(z,u)+\lambda^{\mathrm{T}}(t)\mathrm{f}(z,u)$

Equation (2.7) gives λ + 2u = 0 or λ = – 2u

Equations (2.6) and (2.7) give

$$\lambda_{\mathrm{t}}=2\mathrm{u_t}(x,y,t)=-2z(x,y,t)$$

Hence,

$$Z(x,y,t)=-U_t(x,y,t) \qquad (2.8)$$

Assuming (2.8) as a Fourier solution proposed by Ibiejugba and Onumanyi (1984) and Duchateau and Zachmann(1986).

$$Z(x,y,t)=\sum_{i=1}^{\infty}\alpha_i\,(t)\sin\pi\mathrm{i}\,\mathrm{x}\sin\pi\mathrm{iy} \ (2.9)$$

$$U(x,y,t) = \sum_{i=1}^{\infty} u_i\,(t) \sin \pi i\, x \sin \pi i y \qquad (2.10)$$

This gives the new solution as:

$$Z(x,y,t) = \sum_{i=1}^{\infty} U_{it}\ \sin \pi i\, x \sin \pi i y \qquad (2.11)$$

It then follows that

$$\alpha_i(t) = U_{it}(t)$$

and

$$Z_t(x,y,t) = \sum_{i=1}^{\infty} U_{itt}\,(t) \sin \pi i\, x \sin \pi i y$$

$$Z_{tt}(x,y,t) = \sum_{i=1}^{\infty} U_{itt}\,(t) \sin \pi i\, x \sin \pi i y$$

$$Z_{xx}(x,y,t) = \sum_{i=1}^{\infty} i^2(\pi^2)U_{itt}\,(t) \sin \pi i\, x \sin \pi i y$$

$$Z_{yy}(x,y,t) = \sum_{i=1}^{\infty} i^2(-\pi^2)U_i\,(t) \sin \pi i\, x \sin \pi i y$$

$$Z(x,y,0) = \sum_{i=1}^{\infty} U_{it}\,(0) \sin \pi i\, x \sin \pi i y$$

The constrained equation gives

$$Z_{tt}(x,y,t) + Z_t(x,y,t) = Z_{xx}(x,y,t) + Z_{yy}(x,y,t) + U(x,y,t)$$

Hence

$$\sum_{i=1}^{\infty} U_{ittt}\,(t) \sin \pi i\, x \sin \pi i y + \sum_{i=1}^{\infty} U_{itt}\,(t) \sin \pi i\, x \sin \pi i y =$$

$$\sum_{i=1}^{\infty} -i^2\pi^2 U_{it}\,(t) \sin \pi i\, x \sin \pi i y + \sum_{i=1}^{\infty} -i^2\pi^2 U_{it}\,(t) \sin \pi i\, x \sin \pi i y$$

This implies that

$$U_{ittt}(t) + U_{itt}(t) = -i^2\pi^2 u_{it}(t) - i^2\pi^2 u_{it}(t) + u_i(t)$$

$$U_{ittt}(t) + U_{itt}(t) = -2i^2\pi^2 u_{it}(t) + u_i(t)$$

$$U_{ittt}(t) - U_{itt}(t) = -2i^2\pi^2 u_{it}(t) + u_i(t)$$

and the problem can be written in the form

$$\text{Min}\int_0^1 [u_1^2 + u_2^2 + \cdots \ldots \ldots + u_n{}^2]dt + \int_0^1 \left[u_{it}^2 + u_{2t}^2 + \cdots \ldots \ldots + u_n{}^2\right]dt \qquad (2.12)$$

The corresponding unconstrained problem is given by

$$U_{ittt}(t) - u_{1tt} - 2\pi^2 1^2 u_{it} + u_1$$

$$U_{2ttt}(t) = -u_{2tt} - 2\pi^2 2^2 u_{2t} + u_2$$

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$$U_{nttt}(t) = -U_{ntt} - 2\pi^2 n^2 u_{nt} + u_n \qquad (2.13)$$

3.0 Computational Results

System (2.13) is now solved applying the finite element technique as in Rao (1989) and Bawa (2013). The elements characteristic matrices and vectors are obtained and the overall equations given as

$$[K]U_n^{\rightarrow} = P^{\rightarrow} \qquad (3.1)$$

Where K is the characteristic matrix and $P^{\rightarrow}$ is the characteristic vector. This is solved and the values of the states, controls and cost functional obtained up to ten nodal points as presented on the table of results below.

Table 1: Table of Results

	n = 1			n = 2		
E	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
2	0	0	0	0	0	0
	$0.5454546\pi^2$	$-0.2727273\pi^2$	$0.3719009\pi^4$	$2.1818184\pi^2$	$-1.0909092\pi^2$	$5.950414413\pi^4$
	0	0	0	0	0	0
3	0	0	0	0	0	0
	$0.734694\pi^2$	$-0.244898\pi^2$	$0.599750304\pi^4$	$2.938776\pi^2$	$-0.825864\pi^2$	$9.318416084\pi^4$
	0	$-0.244898\pi^2$	$0.059975030\pi^4$	0	$-0.825864\pi^2$	$0.682051346\pi^4$
	0	0	0	0	0	0
4	0	0	0	0	0	0
	$0.8277864\pi^2$	$-0.2069466\pi^2$	$0.728057219\pi^4$	$3.3111456\pi^2$	$-0.8277864\pi^2$	$11.648915510\pi^4$
	$0.2817376\pi^2$	$-0.277381\pi^2$	$0.156316294\pi^4$	$1.1269504\pi^2$	$-1.109524\pi^2$	$2.501060711\pi^4$
	$-0.2817376\pi^2$	$-0.2069466\pi^2$	$0.122202970\pi^4$	$-1.1269504\pi^2$	$-0.8277864\pi^2$	$1.955247528\pi^4$
	0	0	0	0	0	0
5	0	0	0	0	0	0
	$0.8827475\pi^2$	$-0.1765495\pi^2$	$0.810412800\pi^4$	$3.53099\pi^2$	$-0.706198\pi^2$	$15.379002000\pi^4$
	$0.450321\pi^2$	$-0.2666137\pi^2$	$0.273871800\pi^4$	$1.801284\pi^2$	$-1.0664568\pi^2$	$4.381954155\pi^4$
	0	$-0.2666137\pi^2$	$0.071082800\pi^4$	0	$-1.0664568\pi^2$	$1.133305027\pi^4$
	$-0.450321\pi^2$	$-0.1765495\pi^2$	$0.233958700\pi^4$	$-1.801284\pi^2$	$-0.706198\pi^2$	$3.743341105\pi^4$
	0	0	0	0	0	0
6	0	0	0	0	0	0
	$0.9189222\pi^2$	$-0.1531537\pi^2$	$0.867874065\pi^4$	$3.6756888\pi^2$	$-0.6126148\pi^2$	$13.885985050\pi^4$
	$0.561717\pi^2$	$-0.2467732\pi^2$	$0.315525988\pi^4$	$2.246868\pi^2$	$-0.9870928\pi^2$	$6.022768005\pi^4$
	$0.1889808\pi^2$	$-0.27827\pi^2$	$0.113147935\pi^4$	$0.7559232\pi^2$	$1.11308\pi^2$	$1.810366971\pi^4$
	$-0.1889808\pi^2$	$-0.2467732\pi^2$	$0.096610755\pi^4$	$-0.7559232\pi^2$	$-0.9870928\pi^2$	$1.545772080\pi^4$
	$-0.561717\pi^2$	$-0.1531537\pi^2$	$0.290920864\pi^4$	$-2.246868\pi^2$	$-0.6126148\pi^2$	$5.423712703\pi^4$
	0	0	0	0	0	0

	n = 1			n = 2		
	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
7	0	0	0	0	0	0
	$0.9445051\pi^2$	$-0.1349293\pi^2$	$0.910295799\pi^4$	$3.7780204\pi^2$	$-0.5397172\pi^2$	$14.564732800\pi^4$
	$0.640549\pi^2$	$-0.2264363\pi^2$	$0.461576419\pi^4$	$2.562196\pi^2$	$-0.9057452\pi^2$	$7.385222710\pi^4$
	$0.323565\pi^2$	$-0.2726599\pi^2$	$0.179037859\pi^4$	$1.2942608\pi^2$	$-1.0906396\pi^2$	$2.877195207\pi^4$
	0	$-0.2726599\pi^2$	$0.074342930\pi^4$	0	$-1.0906396\pi^2$	$1.189494737\pi^4$
	$-0.3235652\pi^2$	$-0.2264363\pi^2$	$0.155967836\pi^4$	$-1.2942608\pi^2$	$-0.9057452\pi^2$	$2.495485386\pi^4$
	$-0.640549\pi^2$	$-0.1349293\pi^2$	$0.428508937\pi^4$	$-2.562196\pi^2$	$-0.5397172\pi^2$	$6.651576898\pi^4$
	0	0	0	0	0	0
8	0	0	0	0	0	0
	$0.9635408\pi^2$	$-0.1204426\pi^2$	$0.942917293\pi^4$	$3.861632\pi^2$	$-0.4817704\pi^2$	$15.144304420\pi^4$
	$0.6991744\pi^2$	$-0.2078394\pi^2$	$0.532042057\pi^4$	$2.7966976\pi^2$	$-0.8313576\pi^2$	$8.514230204\pi^4$
	$0.4239104\pi^2$	$-0.2608282\pi^2$	$0.247731377\pi^4$	$1.6956416\pi^2$	$-1.0433128\pi^2$	$3.963702034\pi^4$
	$0.1420416\pi^2$	$-0.2789834\pi^2$	$0.097784526\pi^4$	$0.5681664\pi^2$	$-1.1143336\pi^2$	$1.564549703\pi^4$
	$-0.1420416\pi^2$	$-0.2608282\pi^2$	$0.088207166\pi^4$	$-0.5681664\pi^2$	$-1.04331287\pi^2$	$1.411314803\pi^4$
	$-0.4239104\pi^2$	$-0.2078394\pi^2$	$0.222897243\pi^4$	$-1.6956416\pi^2$	$-0.8313576\pi^2$	$3.566355893\pi^4$
	$-0.6991744\pi^2$	$-0.1204426\pi^2$	$0.503351261\pi^4$	$-2.7966976\pi^2$	$-0.4817704\pi^2$	$8.053620184\pi^4$
	0	0	0	0	0	0
9	0	0	0	0	0	0
	$0.9782514\pi^2$	$-0.1086946\pi^2$	$0.968790317\pi^4$	$3.9130056\pi^2$	$-0.4347784\pi^2$	$15.500645080\pi^4$
	$0.7444332\pi^2$	$-0.1914094\pi^2$	$0.590818347\pi^4$	$2.9777328\pi^2$	$-0.7656376\pi^2$	$9.453092644\pi^4$
	$0.5014431\pi^2$	$-0.2471253\pi^2$	$0.312503961\pi^4$	$2.0057724\pi^2$	$-0.9885012\pi^2$	$5.000257543\pi^4$
	$0.2522754\pi^2$	$-0.2751559\pi^2$	$0.139353646\pi^4$	$1.0091016\pi^2$	$-1.1006236\pi^2$	$2.229658348\pi^4$
	0	$-0.2751559\pi^2$	$0.075710769\pi^4$	0	$-1.1006236\pi^2$	$1.211372309\pi^4$
	$-0.2522754\pi^2$	$-0.2471253\pi^2$	$0.139353646\pi^4$	$-1.0091016\pi^2$	$-0.9885012\pi^2$	$1.995420662\pi^4$
	$-0.5014431\pi^2$	$-0.1914094\pi^2$	$0.288082740\pi^4$	$-2.0057724\pi^2$	$-0.7656376\pi^2$	$4.453952374\pi^4$
	$-0.7444332\pi^2$	$-0.1086946\pi^2$	$0.565995305\pi^4$	$-2.9777328\pi^2$	$-0.4347784\pi^2$	$9.055924885\pi^4$
	0	0	0	0	0	0

10	0	0	0	0	0	0
	$0.9899617\pi^2$	$-9.899617E-02\pi^2$	$0.989822440\pi^4$	$3.959668\pi^2$	$-0.39598468\pi^2$	$15.812120490\pi^4$
	$0.7804113\pi^2$	$-0.1770373\pi^2$	$0.646384000\pi^4$	$3.1216452\pi^2$	$-0.7081492\pi^2$	$10.246144040\pi^4$
	$0.563069\pi^2$	$-0.2333442\pi^2$	$0.371796200\pi^4$	$2.252276\pi^2$	$-0.9333768\pi^2$	$5.943939431\pi^4$
	$0.340106\pi^2$	$-0.2673548\pi^2$	$0.187150600\pi^4$	$1.360424\pi^2$	$-1.0694192\pi^2$	$2.994410885\pi^4$
	$0.113747\pi^2$	$-0.2787295\pi^2$	$0.090628400\pi^4$	$0.4549788\pi^2$	$-1.114918\pi^2$	$1.450047855\pi^4$
	$-0.113747\pi^2$	$-0.2673548\pi^2$	$0.084416800\pi^4$	$-0.4549788\pi^2$	$-1.0694192\pi^2$	$1.350663134\pi^4$
	$-0.340106\pi^2$	$-0.2333442\pi^2$	$0.170121500\pi^4$	$-1.360424\pi^2$	$-0.9333768\pi^2$	$2.721945711\pi^4$
	$-0.563069\pi^2$	$-0.1770373\pi^2$	$0.348388900\pi^4$	$-2.252276\pi^2$	$-0.7081492\pi^2$	$5.780896380\pi^4$
	$-0.7804113\pi^2$	$-9.899616E-02\pi^2$	$0.618842000\pi^4$	$-3.1216452\pi^2$	$-0.39598468\pi^2$	$9.900266550\pi^4$
	0	0	0	0	0	0

	n = 3			n = 4		
E	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
2	0	0	0	0	0	0
	$4.9090914\pi^2$	$-2.4545457\pi^2$	$30.12397\pi^4$	$8.7272736\pi^2$	$-4.3636368\pi^2$	$95.206630610\pi^4$
	0	0	0	0	0	0
3	0	0	0	0	0	0
	$6.612246\pi^2$	$-2.204082\pi^2$	$48.583028400\pi^4$	$11.7515104\pi^2$	$-3.918368\pi^2$	$153.451604500\pi^4$
	0	$-2.204082\pi^2$	$4.857977463\pi^4$	0	$-3.918368\pi^2$	$15.353607780\pi^4$
	0	0	0	0	0	0
4	0	0	0	0	0	0
	$7.4500776\pi^2$	$-1.8625194\pi^2$	$58.972634760\pi^4$	$11.7515104\pi^2$	$-3.311136\pi^2$	$149.061618300\pi^4$
	$2.535638\pi^2$	$-2.496429\pi^2$	$12.661617820\pi^4$	$4.5078016\pi^2$	$-4.438096\pi^2$	$40.016971370\pi^4$
	$-2.535638\pi^2$	$-1.8625194\pi^2$	$9.898438582\pi^4$	$-4.5078016\pi^2$	$-3.311136\pi^2$	$31.283896880\pi^4$
	0	0	0	0	0	0

5	0 $7.9447275\pi^2$ $4.052889\pi^2$ 0 $-4.052889\pi^2$ 0	0 $-1.5889455\pi^2$ $-2.3995233\pi^2$ $-2.3995233\pi^2$ $-1.5889455\pi^2$ 0	0 $65.643443000\pi^4$ $22.183621000\pi^4$ $5.757712100\pi^4$ $18.950657000\pi^4$ 0	0 $14.12396\pi^2$ $7.205136\pi^2$ 0 $-7.205136\pi^2$ 0	0 $-2.824792\pi^2$ $-4.2658192\pi^2$ $-4.2658192\pi^2$ $-2.824792\pi^2$ 0	0 $207.465695900\pi^4$ $70.111198230\pi^4$ $18.197213450\pi^4$ $59.893434620\pi^4$ 0
6	0 $8.2702998\pi^2$ $5.055453\pi^2$ $1.7008272\pi^2$ $-1.7008272\pi^2$ $-5.055453\pi^2$ 0	0 $-1.3783833\pi^2$ $-2.2209588\pi^2$ $-2.50443\pi^2$ $-2.2209588\pi^2$ $-1.3783833\pi^2$ 0	0 $70.922606580\pi^4$ $30.490392730\pi^4$ $9.164982789\pi^4$ $7.825471156\pi^4$ $27.457545560\pi^4$ 0	0 $14.7027552\pi^2$ $8.987472\pi^2$ $3.0236928\pi^2$ $-3.0236928\pi^2$ $-8.987472\pi^2$ 0	0 $-2.4504592\pi^2$ $-3.9483712\pi^2$ $-4.45232\pi^2$ $-3.9483712\pi^2$ $-2.4504592\pi^2$ 0	0 $222.174947700\pi^4$ $96.364288080\pi^4$ $28.965871530\pi^4$ $24.732353280\pi^4$ $86.779403240\pi^4$ 0

	n = 3			n = 4		
	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
7	0 $8.50055459\pi^2$ $5.764941\pi^2$ $2.9120868\pi^2$ 0 $-2.9120868\pi^2$ $-5.764941\pi^2$ 0	0 $-1.2143637\pi^2$ $-2.0379267\pi^2$ $-2.4529391\pi^2$ $-2.459391\pi^2$ $-2.0379267\pi^2$ $-1.2143637\pi^2$ 0	0 $73.734107530\pi^4$ $37.387689970\pi^4$ $14.497159760\pi^4$ $6.048604091\pi^4$ $12.633394770\pi^4$ $34.709223930\pi^4$ 0	0 $15.1120816\pi^2$ $10.248784\pi^2$ $5.1770432\pi^2$ 0 $-5.1770432\pi^2$ $-10.248784\pi^2$ 0	0 $-2.1588688\pi^2$ $-3.6229808\pi^2$ $-4.3625584\pi^2$ $-4.3625584\pi^2$ $-3.6229808\pi^2$ $-2.1588688\pi^2$ 0	0 $233.035724800\pi^4$ $118.163563400\pi^4$ $45.833692090\pi^4$ $19.031915790\pi^4$ $27.189881370\pi^4$ $118.163563400\pi^4$ 0
8	0 $8.6718672\pi^2$ $6.2925696\pi^2$	0 $-1.0839834\pi^2$ $-1.8705546\pi^2$	0 $76.376300750\pi^4$ $43.095738930\pi^4$	0 $15.446528\pi^2$ $11.1867904\pi^2$	0 $-1.9270816\pi^2$ $-3.3254304\pi^2$	0 $242.308870700\pi^4$ $136.202766800\pi^4$

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	$3.8151936\pi^2$	$-2.3474538\pi^2$	$20.066241550\pi^4$	$6.7825664\pi^2$	$-4.1732512\pi^2$	$63.419232550\pi^4$
	$1.2783744\pi^2$	$-2.5072506\pi^2$	$7.920546678\pi^4$	$2.272656\pi^2$	$-4.4573344\pi^2$	$25.032795250\pi^4$
	$-1.2783744\pi^2$	$-2.3474538\pi^2$	$7.144780450\pi^4$	$-2.272656\pi^2$	$-4.1732512\pi^2$	$22.5805545300\pi^4$
	$-3.8151936\pi^2$	$-1.8705546\pi^2$	$18.054676720\pi^4$	$-6.782566\pi^2$	$-3.3254304\pi^2$	$57.061688890\pi^4$
	$-6.2925696\pi^2$	$-1.0839834\pi^2$	$40.771446300\pi^4$	$11.1867904\pi^2$	$-1.9270816\pi^2$	$128.857922900\pi^4$
	0	0	0	0	0	0
9	0	0	0	0	0	0
	$8.8042626\pi^2$	$-0.9782514\pi^2$	$78.472015730\pi^4$	$15.6520224\pi^2$	$-1.7391136\pi^2$	$248.010319200\pi^4$
	$6.6998988\pi^2$	$-1.7226846\pi^2$	$47.856286160\pi^4$	$11.9109312\pi^2$	$-3.0625504\pi^2$	$151.021149100\pi^4$
	$4.5129879\pi^2$	$-2.2241277\pi^2$	$25.313803810\pi^4$	$8.0230896\pi^2$	$-3.9540048\pi^2$	$80.004062920\pi^4$
	$2.2704786\pi^2$	$-2.4764031\pi^2$	$11.287645390\pi^4$	$4.0364064\pi^2$	$-4.4024944\pi^2$	$35.674533570\pi^4$
	0	$-2.476403\pi^2$	$6.132572314\pi^4$	0	$-4.4024944\pi^2$	$19.381956940\pi^4$
	$-2.2704786\pi^2$	$-2.2241277\pi^2$	$10.101817100\pi^4$	$-4.0364064\pi^2$	$-3.9540048\pi^2$	$31.926730580\pi^4$
	$-4.5129879\pi^2$	$-1.7226846\pi^2$	$23.334702020\pi^4$	$-8.0230896\pi^2$	$-3.0625504\pi^2$	$73.749181680\pi^4$
	$-6.6998988\pi^2$	$-0.9782514\pi^2$	$45.845619730\pi^4$	$-11.9109312\pi^2$	$-1.7391136\pi^2$	$145.094514700\pi^4$
	0	0	0	0	0	0
10	0	0	0	0	0	0
	$8.9096553\pi^2$	$-0.8909655\pi^2$	$80.175758000\pi^4$	$15.8393872\pi^2$	$-1.58393872\pi^2$	$253.395048700\pi^4$
	$7.0237017\pi^2$	$-1.5933357\pi^2$	$51.871104000\pi^4$	$12.4865808\pi^2$	$-2.8325968\pi^2$	$158.423667400\pi^4$
	$5.067621\pi^2$	$-2.1000978\pi^2$	$30.091194000\pi^4$	$9.009104\pi^2$	$-3.733472\pi^2$	$95.102768060\pi^4$
	$3.060954\pi^2$	$-2.4061932\pi^2$	$15.159205000\pi^4$	$5.441696\pi^2$	$-4.2776768\pi^2$	$47.910574160\pi^4$
	$1.023723\pi^2$	$-2.5085655\pi^2$	$7.340909700\pi^4$	$1.819952\pi^2$	$-4.459672\pi^2$	$23.200899630\pi^4$
	$-1.023723\pi^2$	$-2.4061932\pi^2$	$6.837774700\pi^4$	$-1.819952\pi^2$	$-4.2776768\pi^2$	$21.610744090\pi^4$
	$-3.060954\pi^2$	$-2.1000978\pi^2$	$13.779850000\pi^4$	$-5.441696\pi^2$	$-3.733472\pi^2$	$43.550868530\pi^4$
	$-5.069621\pi^2$	$-1.5933357\pi^2$	$28.219502000\pi^4$	$-9.009104\pi^2$	$-2.8325965\pi^2$	$89.187559510\pi^4$
	$-7.0237017\pi^2$	$-0.8909655\pi^2$	$50.126187000\pi^4$	$-12.4865808\pi^2$	$-1.58393872\pi^2$	$158.423561900\pi^4$
	0	0	0	0	0	0

	n = 5			n = 6		
E	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
2	0 $13.636365\pi^2$ 0	0 $-6.8181825\pi^2$ 0	0 $232.438063000\pi^4$ 0	0 $19.6363656\pi^2$ 0	0 $-9.8181828\pi^2$ 0	0 $474.659692100\pi^4$ 0
3	0 $18.36735\pi^2$ 0 0	0 $-6.12245\pi^2$ $-6.12245\pi^2$ 0	0 $374.843940000\pi^4$ $37.484390000\pi^4$ 0	0 $26.448984\pi^2$ 0 0	0 $-8.816328\pi^2$ $-8.816328\pi^2$ 0	0 $777.276394000\pi^4$ $77.727639400\pi^4$ 0
4	0 $20.69466\pi^2$ $7.04344\pi^2$ $-7.04344\pi^2$ 0	0 $-5.173665\pi^2$ $-6.934525\pi^2$ $-5173665\pi^2$ 0	0 $455.035762000\pi^4$ $97.697684010\pi^4$ $76.376912910\pi^4$ 0	0 $29.8003104\pi^2$ $10.142532\pi^2$ $-10.142532\pi^2$ 0	0 $-7.450056\pi^2$ $-9.985716\pi^2$ $-7.450056\pi^2$ 0	0 $943.561834300\pi^4$ $202.585479400\pi^4$ $158.374289800\pi^4$ 0
5	0 $22.0686875\pi^2$ $11.258025\pi^2$ 0 $-11.258025\pi^2$ 0	0 $-4.4137375\pi^2$ $-6.6653425\pi^2$ $-4.4137375\pi^2$ 0	0 $506.508046700\pi^4$ $171.169917500\pi^4$ $146.224205600\pi^4$ 0	0 $31.77891\pi^2$ $16.211556\pi^2$ 0 $-16.211556\pi^2$ 0	0 $-6.355782\pi^2$ $-9.5908932\pi^2$ $-9.5980932\pi^2$ $-6.355782\pi^2$ 0	0 $1050.295086000\pi^4$ $354.937941000\pi^4$ $92.123393080\pi^4$ $303.210512800\pi^4$ 0
6	0 $22.97452\pi^2$ $14.042925\pi^2$ $4.72452\pi^2$ $-4.72452\pi^2$ $-14.042925\pi^2$ 0	0 $-3.8288425\pi^2$ $-6.16933\pi^2$ $-6.95675\pi^2$ $-6.95675\pi^2$ $-6.16933\pi^2$ 0	0 $528.515549100\pi^4$ $235.442058200\pi^4$ $70.717459790\pi^4$ $70.717459790\pi^4$ $235.264375200\pi^4$ 0	0 $33.0811992\pi^2$ $20.221812\pi^2$ $6.8033088\pi^2$ $-6.8033088\pi^2$ $-20.221812\pi^2$ 0	0 $-5.515533\pi^2$ $-8.8838352\pi^2$ $-10.01772\pi^2$ $-8.8838352\pi^2$ $-5.513533\pi^2$ 0	0 $1124.764787000\pi^4$ $487.844208400\pi^4$ $146.639724600\pi^4$ $125.207538500\pi^4$ $439.320726700\pi^4$ 0

	n = 5			n = 6		
	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
7	0	0	0	0	0	0
	$23.6126275\pi^2$	$-3.3732325\pi^2$	$568.934875000\pi^4$	$34.0021836\pi^2$	$-4.8574548\pi^2$	$1179.743357000\pi^4$
	$16.013725\pi^2$	$-5.6609075\pi^2$	$288.485262100\pi^4$	$23.059764\pi^2$	$-8.1517068\pi^2$	$598.203039500\pi^4$
	$8.08913\pi^2$	$-6.8164975\pi^2$	$111.898662300\pi^4$	$11.6483472\pi^2$	$-9.8157564\pi^2$	$232.033066200\pi^4$
	0	$-6.8164975\pi^2$	$46.464638170\pi^4$	0	$-9.8157564\pi^2$	$96.349073700\pi^4$
	$-8.08913\pi^2$	$-5.6609075\pi^2$	$97.479897880\pi^4$	$-11.6483472\pi^2$	$-8.1517068\pi^2$	$202.13431620\pi^4$
	$-16.013725\pi^2$	$-3.3732325\pi^2$	$267.818085900\pi^4$	$-23.059764\pi^2$	$-4.8574548\pi^2$	$555.347582900\pi^4$
	0	0	0	0	0	0
8	0	0	0	0	0	0
	$24.1352\pi^2$	$-3.011065\pi^2$	$591.574391500\pi^4$	$34.754688\pi^2$	$-4.3359336\pi^2$	$1226.688658000\pi^4$
	$17.47936\pi^2$	$-5.195985\pi^2$	$332.526286100\pi^4$	$25.1702784\pi^2$	$-7.4822184\pi^2$	$689.526506900\pi^4$
	$10.59776\pi^2$	$-6.520705\pi^2$	$154.832110700\pi^4$	$15.2607744\pi^2$	$-9.3898152\pi^2$	$321.059864800\pi^4$
	$3.55104\pi^2$	$-6.964585\pi^2$	$61.115329300\pi^4$	$5.1134976\pi^2$	$-10.0290024\pi^2$	$126.728746800\pi^4$
	$-3.55104\pi^2$	$-6.520705\pi^2$	$55.129478780\pi^4$	$-5.1134976\pi^2$	$-9.3898152\pi^2$	$114.316487200\pi^4$
	$-10.59776\pi^2$	$-5.195985\pi^2$	$139.310777100\pi^4$	$-15.2607744\pi^2$	$-7.4822184\pi^2$	$288.874827500\pi^4$
	$-17.47936\pi^2$	$-3.011065\pi^2$	$314.594538400\pi^4$	$-25.1702784\pi^2$	$-4.3359336\pi^2$	$652.343234900\pi^4$
	0	0	0	0	0	0
9	0	0	0	0	0	0
	$24.456285\pi^2$	$-2.717365\pi^2$	$605.493704000\pi^4$	$35.2170504\pi^2$	$-3.9130056\pi^2$	$1255.552247000\pi^4$
	$18.61083\pi^2$	$-4.785235\pi^2$	$369.261467300\pi^4$	$26.7995952\pi^2$	$-6.8907384\pi^2$	$765.700578600\pi^4$
	$12.5360775\pi^2$	$-6.1781325\pi^2$	$195.322560300\pi^4$	$18.0519516\pi^2$	$-8.8965108\pi^2$	$404.995902900\pi^4$
	$6.306885\pi^2$	$-6.8788975\pi^2$	$87.096029220\pi^4$	$9.0819144\pi^2$	$-9.9056124\pi^2$	$180.602326200\pi^4$
	0	$-6.8788975\pi^2$	$47.319230820\pi^4$	0	$-9.9056124\pi^2$	$98.121157050\pi^4$
	$-6.306885\pi^2$	$-6.1781325\pi^2$	$77.946119590\pi^4$	$-9.0819144\pi^2$	$-8.8965108\pi^2$	$161.629073600\pi^4$
	$-12.5360775\pi^2$	$-4.785235\pi^2$	$180.051713100\pi^4$	$-18.0519516\pi^2$	$-6.8907384\pi^2$	$373.355232300\pi^4$

	$-18.61083\pi^2$	$-2.717365\pi^2$	$353.747065800\pi^4$	$-26.7995952\pi^2$	$-3.9130056\pi^2$	$733.529915700\pi^4$
	0	0	0	0	0	0
	0	0	0	0	0	0
	$24.7490425\pi^2$	$-2.47490425\pi^2$	$618.640255700\pi^4$	$35.6386212\pi^2$	$-3.56386212\pi^2$	$1282.812434000\pi^4$
	$195102825\pi^2$	$-4.4259325\pi^2$	$400.240001700\pi^4$	$28.0948068\pi^2$	$-6.3733428\pi^2$	$829.937667600\pi^4$
	$14.076725\pi^2$	$-5.83355\pi^2$	$232.184492300\pi^4$	$20.270484\pi^2$	$-8.4003912\pi^2$	$481.459093900\pi^4$
	$8.50265\pi^2$	$-6.68387\pi^2$	$166.969175200\pi^4$	$12.243816\pi^2$	$-9.6247728\pi^2$	$242.5464193\pi^4$
10	$2.843675\pi^2$	$-6.9682375\pi^2$	$56.642821360\pi^4$	$4.094892\pi^2$	$-10.034262\pi^2$	$117.454554400\pi^4$
	$-2.843675\pi^2$	$-6.68387\pi^2$	$52.760605680\pi^4$	$-4.094892\pi^2$	$-9.6247728\pi^2$	$109.404391900\pi^4$
	-8.50265	$-5.83355\pi^2$	$106.325362600\pi^4$	$-12.243816\pi^2$	$-8.4003912\pi^2$	$220.477602600\pi^4$
	$-4.04672\pi^2$	$-4.4259325\pi^2$	$217.757846100\pi^4$	$-20.270484\pi^2$	$-6.3733428\pi^2$	$451.512020000\pi^4$
	$19.5102825\pi^2$	$-2.47490425\pi^2$	$386.776274300\pi^4$	$-28.0948068\pi^2$	$-3.56386212\pi^2$	$802.019282300\pi^4$
	0	0	0	0	0	0

	n = 7			n = 8		
E	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
	0	0	0	0	0	0
2	$26.7272754\pi^2$	$-13.3636377\pi^2$	$892.934062900\pi^4$	$34.9090944\pi^2$	$-17.4545472\pi^2$	$1523.306090000\pi^4$
	0	0	0	0	0	0
	0	0	0	0	0	0
3	$36.000006\pi^2$	$-12.000002\pi^2$	$1440.000480000\pi^4$	$47.020416\pi^2$	$-15.673472\pi^2$	$2458.500934000\pi^4$
	0	$-12.000002\pi^2$	$1440.000480000\pi^4$	0	$-15.673472\pi^2$	$245.657724500\pi^4$
	0	0	0	0	0	0
	0	0	0	0	0	0
	$405615336\pi^2$	$-10.1403834\pi^2$	$1748.065383000\pi^4$	$52.9783296\pi^2$	$-13.2445824\pi^2$	$2982.122370000\pi^4$
4	$13.8051424\pi^2$	$-13.591669\pi^2$	$373.315422900\pi^4$	$18.0312064\pi^2$	$-17.752384\pi^2$	$640.271541900\pi^4$
	$-13.8051424\pi^2$	$-10.1403834\pi^2$	$293.409332200\pi^4$	$-18.0312064\pi^2$	$-13.2445824\pi^2$	$500.543367200\pi^4$
	0	0	0	0	0	0

5	0	0	0	0	0	0
	$43.2546275\pi^2$	$-8.6509255\pi^2$	$1945.801312000\pi^4$	$56.495536\pi^2$	$-11.299168\pi^2$	$3319.416785000\pi^4$
	$22.065729\pi^2$	$-13.0640713\pi^2$	$657.566355200\pi^4$	$28.820544\pi^2$	$-17.0632768\pi^2$	$1121.779172000\pi^4$
	0	$-13.0640713\pi^2$	$170.669958900\pi^4$	0	$-17.0632768\pi^2$	$291.155415200\pi^4$
	$-22.065729\pi^2$	$-8.6509255\pi^2$	$561.734908300\pi^4$	$-28.820544\pi^2$	$-11.299168\pi^2$	$958.125793700\pi^4$
	0	0	0	0	0	0
6	0	0	0	0	0	0
	$45.0271878\pi^2$	$-7.5045313\pi^2$	$2083.765631000\pi^4$	$58.8110208\pi^2$	$-9.8018368\pi^2$	$3554.812172000\pi^4$
	$27.524133\pi^2$	$-12.0918868\pi^2$	$903.791623800\pi^4$	$35.949888\pi^2$	$-15.7934848\pi^2$	$1541.808609000\pi^4$
	$9.2600592\pi^2$	$-13.63523\pi^2$	$271.668193500\pi^4$	$12.0947712\pi^2$	$-17.80929\pi^2$	$463.454300700\pi^4$
	$-9.2600592\pi^2$	$-12.0918868\pi^2$	$231.962422800\pi^4$	$-12.947712\pi^2$	$-15.7934848\pi^2$	$417.077408200\pi^4$
	$-27.524133\pi^2$	$-7.5045313\pi^2$	$813.895887400\pi^4$	$-35.949888\pi^2$	$-9.8018368\pi^2$	$1388.470452000\pi^4$
	0	0	0	0	0	0

	n = 7			n = 8		
	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
7	0	0	0	0	0	0
	$46.2807999\pi^2$	$-6.6115357\pi^2$	$2185.620216000\pi^4$	$60.4483264\pi^2$	$8.6354752\pi^2$	$3728.571596000\pi^4$
	$31.386901\pi^2$	$-11.0953787\pi^2$	$1108.244983000\pi^4$	$40.9951136\pi^2$	$-14.4919232\pi^2$	$1890.615177000\pi^4$
	$15.8546948\pi^2$	$-13.3603351\pi^2$	$429.869901200\pi^4$	$20.7087728\pi^2$	$-17.4202336\pi^2$	$732.292959400\pi^4$
	0	$-13.3603351\pi^2$	$178.498978800\pi^4$	0	$-17.4202336\pi^2$	$303.464538700\pi^4$
	$-15.8546948\pi^2$	$-11.0953787\pi^2$	$374.478775700\pi^4$	$-20.7081728\pi^2$	$-14.4919232\pi^2$	$638.844224000\pi^4$
	$-31.386901\pi^2$	$-6.6115357\pi^2$	$1028.849954000\pi^4$	$-40.9951136\pi^2$	$-8.6354752\pi^2$	$1775.170771000\pi^4$
	0	0	0	0	0	0
8	0	0	0	0	0	0
	$47.304992\pi^2$	$-5.9016874\pi^2$	$2272.592182000\pi^4$	$61.786112\pi^2$	$-7.7083264\pi^2$	$3876.941932000\pi^4$
	$34.2595456\pi^2$	$-10.1841306\pi^2$	$1277.432981000\pi^4$	$44.7471616\pi^2$	$-13.3017216\pi^2$	$2179.244269000\pi^4$

	$20.776096\pi^2$	$-12.7805818\pi^2$	$594.780303650\pi^4$	$27.1302656\pi^2$	$-16.6930048\pi^2$	$1014.707721000\pi^4$
	$6.9600384\pi^2$	$-13.6505860\pi^2$	$234.785405700\pi^4$	$9.0906624\pi^2$	$-17.8293376\pi^2$	$400.525422100\pi^4$
	$-6.9600384\pi^2$	$-12.7805818\pi^2$	$211.785405700\pi^4$	$-9.0906624\pi^2$	$-16.6930048\pi^2$	$361.296552100\pi^4$
	$-20.7716096\pi^2$	$10.1841306\pi^2$	$535.176281500\pi^4$	$-27.1302656\pi^2$	$-13.3017216\pi^2$	912.987093100
	$-34.2595456\pi^2$	$5.9016574\pi^2$	$1208.546379000\pi^4$	$-44.7471616\pi^2$	$-7.7083264\pi^2$	$2061.726713000\pi^4$
	0	0	0	0	0	0
9	0	0	0	0	0	0
	$47.9343186\pi^2$	$-5.3260354\pi^2$	$2326.065553000\pi^4$	$62.608064\pi^2$	$-6.9564544\pi^2$	$3902.398121000\pi^4$
	$36.4772268\pi^2$	$-9.3790606\pi^2$	$1418.554853000\pi^4$	$47.6437248\pi^2$	$-12.2502016\pi^2$	$2419.992407000\pi^4$
	$24.5707119\pi^2$	$-12.1091397\pi^2$	$750.351147500\pi^4$	$32.0923584\pi^2$	$-15.8160192\pi^2$	$1280.065418000\pi^4$
	$12.3614946\pi^2$	$-13.4826391\pi^2$	$334.588105800\pi^4$	$16.1456256\pi^2$	$-17.6099776\pi^2$	$570.792537100\pi^4$
	0	$-13.4826391\pi^2$	$181.781557100\pi^4$	0	$-17.6099776\pi^2$	$310.111311100\pi^4$
	$-12.3614946\pi^2$	$-12.1091397\pi^2$	$299.437813000\pi^4$	$-16.1456256\pi^2$	$-15.8160192\pi^2$	$570.792537100\pi^4$
	$-24.5707119\pi^2$	$-9.3790606\pi^2$	$750.354475000\pi^4$	$-32.0923584\pi^2$	$-12.2502016\pi^2$	$1179.986907000\pi^4$
	$-36.4772268\pi^2$	$-5.3260354\pi^2$	$1418.554853000\pi^4$	$-47.6437248\pi^2$	$-6.9564544\pi^2$	$2318.316765000\pi^4$
	$-47.9343186\pi^2$	0	$2326.065553000\pi^4$	0	0	0
	0		0			
10	0	0	0	0	0	0
	$48.5081233\pi^2$	$-4.85081233\pi^2$	$2376.568406000\pi^4$	$63.3575488\pi^2$	$-6.33575488\pi^2$	$4054.597417000\pi^4$
	$38.2401537\pi^2$	$-8.6748277\pi^2$	$1537.561991000\pi^4$	$49.9463232\pi^2$	$-11.3303872\pi^2$	$2623.012875000\pi^4$
	$27.590381\pi^2$	$-11.4338658\pi^2$	$891.962410900\pi^4$	$36.036416\pi^2$	$-14.9340288\pi^2$	$1521.520565000\pi^4$
	$16.665194\pi^2$	$-13.1003852\pi^2$	$449.348783400\pi^4$	$21.766784\pi^2$	$-17.1107072\pi^2$	$766.569186600\pi^4$
	$5.573603\pi^2$	$-13.6577455\pi^2$	$217.599062500\pi^4$	$7.279808\pi^2$	$-17.838688\pi^2$	$371.221244200\pi^4$
	$-5.573603\pi^2$	$-13.1003852\pi^2$	$202.685142800\pi^4$	$-7.279808\pi^2$	$-17.1107072\pi^2$	$345.771905400\pi^4$
	$-16.665194\pi^2$	$-11.4338658\pi^2$	$408.020392900\pi^4$	$-21.766784\pi^2$	$-14.9340288\pi^2$	$696.818078000\pi^4$
	$-27.590381\pi^2$	$-8.6748277\pi^2$	$836.481759300\pi^4$	$-36.036416\pi^2$	$-11.3303872\pi^2$	$1427.000952000\pi^4$
	$-38.2401537\pi^2$	$-4.85081233\pi^2$	$1485.839735000\pi^4$	$-49.9463232\pi^2$	$-6.33575488\pi^2$	$2535.053628000\pi^4$
	0	0	0	0	0	0

	n = 9			n = 10		
E	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
2	0	0	0	0	0	0
	$44.1818226\pi^2$	$-22.0909113\pi^2$	$3578.727631\pi^4$	$54.54546\pi^2$	$-27.2727\pi^2$	$3719.007372\pi^4$
	0	0	0	0	0	0
3	0	0	0	0	0	0
	$59.510214\pi^2$	$-19.836738\pi^2$	$4820.327334000\pi^4$	$73.4694\pi^2$	$-24.4898\pi^2$	$5997.503040000\pi^4$
	0	$-19.836738\pi^2$	$393.498078800\pi^4$	0	$-24.4898\pi^2$	$599.750304000\pi^4$
	0	0	0	0	0	0
4	0	0	0	0	0	0
	$67.0506984\pi^2$	$-16.7626746\pi^2$	$4776.783416000\pi^4$	$82.77864\pi^2$	$-20.69466\pi^2$	$7280.572193000\pi^4$
	$22.820697\pi^2$	$-22.46786\pi^2$	$526.874544500\pi^4$	$28.17376\pi^2$	$-27.7381\pi^2$	$1563.162944000\pi^4$
	$-22.820697\pi^2$	$-16.7626746\pi^2$	$801.771451200\pi^4$	$-28.17376\pi^2$	$-20.69466\pi^2$	$1222.029705000\pi^4$
	0	0	0	0	0	0
5	0	0	0	0	0	0
	$71.5025475\pi^2$	$-14.3005095\pi^2$	$5317.118871000\pi^4$	$88.2747\pi^2$	$-17.65495\pi^2$	$8104.119920000\pi^4$
	$36.476001\pi^2$	$-21.5957097\pi^2$	$1796.873326000\pi^4$	$45.0321\pi^2$	$-26.66137\pi^2$	$2738.718681000\pi^4$
	0	$-21.5957097\pi^2$	$457.776393600\pi^4$	0	$-26.66137\pi^2$	$710.828650300\pi^4$
	$-36.476001\pi^2$	$-14.3005095\pi^2$	$1535.003221000\pi^4$	$-45.0321\pi^2$	$-17.65495\pi^2$	$2339.587290000\pi^4$
	0	0	0	0	0	0
6	0	0	0	0	0	0
	$74.4326982\pi^2$	$-12.4054497\pi^2$	$5665.876971000\pi^4$	$91.8222\pi^2$	$-15.31537\pi^2$	$8665.876971000\pi^4$
	$45.499077\pi^2$	$-19.9886292\pi^2$	$1665.876431000\pi^4$	$56.1717\pi^2$	$-24.67732\pi^2$	$5764.230003000\pi^4$
	$15.3074448\pi^2$	$-22.53987\pi^2$	$1131.479357000\pi^4$	$18.89808\pi^2$	$-27.827\pi^2$	$1131.479357000\pi^4$
	$-15.3074448\pi^2$	$-19.9886292\pi^2$	$101.519357000\pi^4$	$-18.89808\pi^2$	$-24.67732\pi^2$	$966.107550100\pi^4$
	$-45.499077\pi^2$	$-12.405447\pi^2$	$1231.24157000\pi^4$	$-56.1717\pi^2$	$-15.31537\pi^2$	$3389.820439000\pi^4$
	0	0	0	0	0	0

n = 10	n = 9

	Z_{n_e}	U_{n_e}	J_{n_e}	Z_{n_e}	U_{n_e}	J_{n_e}
7	0	0	0	0	0	0
	$94.45051\pi^2$	$-13.49293\pi^2$	$9102.957190000\pi^4$	$76.5049131\pi^2$	$-10.929249\pi^2$	$5972.450212000\pi^4$
	$64.0549\pi^2$	$-22.64363\pi^2$	$4615.764194000\pi^4$	$51.884469\pi^2$	$-18.3413403\pi^2$	$3028.402887000\pi^4$
	$32.35652\pi^2$	$-27.26599\pi^2$	$1790.378597000\pi^4$	$26.2087812\pi^2$	$-22.0854519\pi^2$	$1174.667398000\pi^4$
	0	$-27.26599\pi^2$	$743.434210700\pi^4$	0	$-22.0854519\pi^2$	$487.767185600\pi^4$
	$-32.35652\pi^2$	$-22.64363\pi^2$	$1559.678366000\pi^4$	$-26.2087812\pi^2$	$-18.3413403\pi^2$	$1023.304976000\pi^4$
	$-64.0549\pi^2$	$-13.49293\pi^2$	$4285.072373000\pi^4$	$-51.884469\pi^2$	$-10.929249\pi^2$	$2811.446607000\pi^4$
	0	0	0	0	0	0
8	0	0	0	0	0	0
	$96.35408\pi^2$	$-12.04426\pi^2$	$9429.172932000\pi^4$	$78.198048\pi^2$	$-9.7558506\pi^2$	$6210.110345000\pi^4$
	$69.91744\pi^2$	$-20.7839\pi^2$	$5320.420578000\pi^4$	$56.6331264\pi^2$	$-16.8349914\pi^2$	$3490.516998000\pi^4$
	$42.39104\pi^2$	$-26.08282\pi^2$	$2477.313771000\pi^4$	$34.3367424\pi^2$	$-16.870842\pi^2$	$1463.637188000\pi^4$
	$14.20416\pi^2$	$-27.85834\pi^2$	$977.845268900\pi^4$	$11.505321\pi^2$	$-22.5652554\pi^2$	$641.563165600\pi^4$
	$-14.20416\pi^2$	$-26.08282\pi^2$	$882.071660500\pi^4$	$-11.505321\pi^2$	$-16.870842\pi^2$	$416.997721100\pi^4$
	$-42.39104\pi^2$	$-20.7839\pi^2$	$2228.972434000\pi^4$	$-34.3367424\pi^2$	$-16.8349914\pi^2$	$1462.931480000\pi^4$
	$-69.91744\pi^2$	$-12.04426\pi^2$	$243.419815100\pi^4$	$-56.6331264\pi^2$	$-9.7558506\pi^2$	$3302.487355000\pi^4$
	0	0	0	0	0	0
9	0	0	0	0	0	0
	$97.82514\pi^2$	$-10.86946\pi^2$	$9687.903177000\pi^4$	$79.2383634\pi^2$	$-8.804214\pi^2$	$6356.232418000\pi^4$
	$74.44332\pi^2$	$-19.14094\pi^2$	$5908.183477000\pi^4$	$60.2990892\pi^2$	$-15.5041614\pi^2$	$3876.454526000\pi^4$
	$50.14431\pi^2$	$-24.71253\pi^2$	$3125.160964000\pi^4$	$40.6168911\pi^2$	$-20.0171493\pi^2$	$2050.418109000\pi^4$
	$25.22754\pi^2$	$-27.51559\pi^2$	$1393.536467000\pi^4$	$20.4343074\pi^2$	$-22.2876279\pi^2$	$914.239150600\pi^4$
	0	$-27.51559\pi^2$	$1393.536467000\pi^4$	0	$-22.2876279\pi^2$	$496.722805200\pi^4$
	$-25.22754\pi^2$	$-24.71253\pi^2$	$2880.827429000\pi^4$	$-20.4343074\pi^2$	$-20.0171493\pi^2$	$818.247185000\pi^4$
	$-50.14431\pi^2$	$-19.14094\pi^2$	$3659.953053000\pi^4$	$-40.6168911\pi^2$	$-15.5041614\pi^2$	$1890.110863000\pi^4$
	$-74.44332\pi^2$	$-10.86946\pi^2$	0	$-60.2990892\pi^2$	$-8.804214\pi^2$	$3713.494343000\pi^4$
	0	0		0	0	0

10	0	0	0	0	0	0
	$98.99617\pi^2$	$-9.899617\pi^2$	$9898.244091000\pi^4$	$80.186841\pi^2$	$-8.01868977\pi^2$	$6494.232190000\pi^4$
	$78.04113\pi^2$	$-17.70373\pi^2$	$6403.840028000\pi^4$	$63.2133153\pi^2$	$-14.3400213\pi^2$	$4201.559442000\pi^4$
	$56.3069\pi^2$	$-23.33442\pi^2$	$3715.418661000\pi^4$	$45.61029\pi^2$	$-18.9008802\pi^2$	$2437.541826000\pi^4$
	$34.0106\pi^2$	$-26.73548\pi^2$	$1871.502526000\pi^4$	$27.548586\pi^2$	$-21.6557388\pi^2$	$1227.895614000\pi^4$
	$11.3747\pi^2$	$-27.87295\pi^2$	$906.285141800\pi^4$	$9.213507\pi^2$	$-22.5770895\pi^2$	$594.613681500\pi^4$
	$-11.3747\pi^2$	$-26.73548\pi^2$	$1871.506803000\pi^4$	$-9.213507\pi^2$	$-21.6557388\pi^2$	$106.544450000\pi^4$
	$-34.0106\pi^2$	$-23.33442\pi^2$	$1701.216069000\pi^4$	$-27.548586\pi^2$	$-18.9008802\pi^2$	$1116.167863000\pi^4$
	$-56.3069\pi^2$	$-17.70373\pi^2$	$3483.889044000\pi^4$	$45.61029\pi^2$	$-14.3400213\pi^2$	$2285.934765000\pi^4$
	$-78.04113\pi^2$	$-9.899617\pi^2$	$6188.420388000\pi^4$	$63.2133153\pi^2$	$-8.01868977\pi^2$	$4060.222604000\pi^4$
	0	0	0	0	0	0

Conclusion

The results for the optimal state, optimal control and the optimal cost functional for the two dimensional energized wave equation were obtained up to ten nodal points. Following Singh and Titli(1978), the objective functional and the dynamical energized wave are penalized to obtain the Hamiltonian for the system leading to the corresponding unconstrained problem in which the Finite Element Method was applied. It is observed that as the strata profiles in space increase from n=1 to n=10, the energy level gets higher, which symbolizes a high attendance of frequency observations and high cost functional.

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