Improved Continuous Block Integrator for Direct Solution of First-Order Ordinary Differential Equations

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Abstract

In this paper, we propose an improved continuous block integrator for the solution of first-order ordinary differential equations. The approximate solution used in deriving the integrator is a combination of power series and exponential function. The basic properties of the integrator was investigated and found to be zero-stable, consistent and convergent. The efficiency of the integrator was tested on some numerical examples and was found to give better approximation than the existing integrators we compared result with.

Keywords: Approximate Solution, Block Integrator, Exponential Function, Order, Power Series
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Introduction

In recent times, the integration of Ordinary Differential Equations (ODEs) is carried out using some kinds of block methods. Therefore, in this paper, we propose an improved continuous block integrator for the solution of first-order ODEs of the form:

$$y' = f(x,y),\ y(a)=\eta\ \forall\, a \le x \le b \quad \ldots\ldots\ldots(1)$$

Where: f is continuous within the interval of integration $[a,b]$. We assume that f satisfies Lipchitz condition which guarantees the existence and uniqueness of solution of (1). The problem (1) occurs mainly in the study of dynamical systems and electrical networks. According to Sunday (2011) and Kandasamy *et al.* (2005), equation (1) is used in simulating the growth of populations, trajectory of a particle, simple harmonic motion, deflection of a beam, etc. It is also important to note that mixture models, SIR model and other similar models can be written in the form (1).

Development of Linear Multistep Methods (LMMs) for solving ODEs can be generated using methods such as Taylor's series, numerical integration and collocation methods, which are restricted by an assumed order of convergence, Ehigie *et al.* (2010). In this work, we will follow suite from the previous paper of Sunday *et al.* (2013) by deriving an improved continuous block integrator in a multistep collocation technique.

Various authors proposed LMMs to generate numerical solution to (1), please refer to Butcher (2003), Zarina *et al.* (2005), Awoyemi *et al.* (2007), Chollom and Zirra

(2009), Umar and Yahaya (2010), Areo *et al*. (2011), Badmus and Mishelia (2011), Ibijola et al. (2011), Chollom *et al.* (2012), Odekunle *et al.* (2012A, 2012B), among others. These authors proposed integrators in which the approximate solution ranges from power series, Chebychev's, Lagrange's and Laguerre's polynomials. The advantages of LMMs over single step methods have been extensively discussed in Awoyemi *et al.* (2007).

In this paper, we propose an improved continuous block integrator, in which the approximate solution is the combination of power series and exponential function. This work is an improvement on Sunday *et al.* (2013).

Methodology

Construction of the Improved Block Integrator

To derive this integrator, interpolation and collocation procedures are used by choosing interpolation point s at a grid point and collocation points r at all points giving rise to $\xi = s + r$ system of equations whose coefficients are determined by using appropriate procedures. The approximate solution to (1) is taken to be a combination of power series and exponential function given by:

$$y(x) = \sum_{j=0}^{6} a_j x^j + a_7 \sum_{j=0}^{7} \frac{\alpha^j x^j}{j!} \qquad (2)$$

with the first derivative given by:

$$y'(x) = \sum_{j=0}^{6} j a_j x^{j-1} + a_7 \sum_{j=1}^{7} \frac{\alpha^j x^{j-1}}{(j-1)!} \qquad (3)$$

where $a_j, \alpha^j \in \mathbb{R}$ for $j = 0(1)7$ and $y(x)$ is continuously differentiable. Let the solution of (1) be sought on the partition $\pi_N : a = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < \ldots < x_N = b$, of the integration interval $[a,b]$ with a constant step-size h, given by, $h = x_{n+1} - x_n$, $n = 0,1,\ldots,N$.

Then, substituting (3) in (1) gives:

$$f(x,y) = \sum_{j=0}^{6} j a_j x^{j-1} + a_7 \sum_{j=1}^{7} \frac{\alpha^j x^{j-1}}{(j-1)!} \qquad (4)$$

Now, interpolating (2) at point $x_{n+s}, s = 0$ and collocating (4) at points $x_{n+r}, r = 0(1)6$, leads to the following system of equations:

$$AX = U \qquad (5)$$

where

$$A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7]^T$$
$$U = [y_n, f_n, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}]^T$$

and

$$X=\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & \left(1+\alpha x_n+\frac{\alpha^2 x_n^2}{2!}+\frac{\alpha^3 x_n^3}{3!}+\frac{\alpha^4 x_n^4}{4!}+\frac{\alpha^5 x_n^5}{5!}+\frac{\alpha^6 x_n^6}{6!}+\frac{\alpha^7 x_n^7}{7!}\right) \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & \left(\alpha+\alpha^2 x_n+\frac{\alpha^3 x_n^2}{2!}+\frac{\alpha^4 x_n^3}{3!}+\frac{\alpha^5 x_n^4}{4!}+\frac{\alpha^6 x_n^5}{5!}+\frac{\alpha^7 x_n^6}{6!}\right) \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & \left(\alpha+\alpha^2 x_{n+1}+\frac{\alpha^3 x_{n+1}^2}{2!}+\frac{\alpha^4 x_{n+1}^3}{3!}+\frac{\alpha^5 x_{n+1}^4}{4!}+\frac{\alpha^6 x_{n+1}^5}{5!}+\frac{\alpha^7 x_{n+1}^6}{6!}\right) \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & \left(\alpha+\alpha^2 x_{n+2}+\frac{\alpha^3 x_{n+2}^2}{2!}+\frac{\alpha^4 x_{n+2}^3}{3!}+\frac{\alpha^5 x_{n+2}^4}{4!}+\frac{\alpha^6 x_{n+2}^5}{5!}+\frac{\alpha^7 x_{n+2}^6}{6!}\right) \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & \left(\alpha+\alpha^2 x_{n+3}+\frac{\alpha^3 x_{n+3}^2}{2!}+\frac{\alpha^4 x_{n+3}^3}{3!}+\frac{\alpha^5 x_{n+3}^4}{4!}+\frac{\alpha^6 x_{n+3}^5}{5!}+\frac{\alpha^7 x_{n+3}^6}{6!}\right) \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & \left(\alpha+\alpha^2 x_{n+4}+\frac{\alpha^3 x_{n+4}^2}{2!}+\frac{\alpha^4 x_{n+4}^3}{3!}+\frac{\alpha^5 x_{n+4}^4}{4!}+\frac{\alpha^6 x_{n+4}^5}{5!}+\frac{\alpha^7 x_{n+4}^6}{6!}\right) \\ 0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 & \left(\alpha+\alpha^2 x_{n+5}+\frac{\alpha^3 x_{n+5}^2}{2!}+\frac{\alpha^4 x_{n+5}^3}{3!}+\frac{\alpha^5 x_{n+5}^4}{4!}+\frac{\alpha^6 x_{n+5}^5}{5!}+\frac{\alpha^7 x_{n+5}^6}{6!}\right) \\ 0 & 1 & 2x_{n+6} & 3x_{n+6}^2 & 4x_{n+6}^3 & 5x_{n+6}^4 & 6x_{n+6}^5 & \left(\alpha+\alpha^2 x_{n+6}+\frac{\alpha^3 x_{n+6}^2}{2!}+\frac{\alpha^4 x_{n+6}^3}{3!}+\frac{\alpha^5 x_{n+6}^4}{4!}+\frac{\alpha^6 x_{n+6}^5}{5!}+\frac{\alpha^7 x_{n+6}^6}{6!}\right) \end{bmatrix}$$

Solving (5), for $a_j\text{'s}, j=0(1)7$ and substituting back into (2) gives a continuous linear multistep method of the form:

$$y(x)=\alpha_0(x)y_n+h\sum_{j=0}^{6}\beta_j(x)f_{n+j} \quad \ldots\ldots\ldots(6)$$

Where: the coefficients of y_{n+1} and f_{n+j} gives

:

$$\left.\begin{aligned}
\alpha_0 &= 1 \\
\beta_0 &= \frac{1}{60480}(12t^7 - 294t^6 + 294t^5 - 15435t^4 + 45472t^3 - 74088t^2 + 60480t) \\
\beta_1 &= -\frac{1}{2520}(3t^7 - 70t^6 + 651t^5 - 3045t^4 + 7308t^3 - 7560t^2) \\
\beta_2 &= -\frac{1}{2520}(3t^7 - 70t^6 + 651t^5 - 3045t^4 + 7308t^3 - 7560t^2) \\
\beta_3 &= -\frac{1}{3780}(15t^7 - 315t^6 + 2541t^5 - 9765t^4 + 17780t^3 - 12600t^2) \\
\beta_4 &= \frac{1}{20160}(60t^7 - 1190t^6 + 8988t^5 - 32235t^4 + 55440t^3 - 37800t^2) \\
\beta_5 &= -\frac{1}{2520}(3t^7 - 56t^6 + 399t^5 - 1365t^4 + 2268t^3 - 1512t^2) \\
\beta_6 &= \frac{1}{60480}(12t^7 - 210t^6 + 1428t^5 - 4725t^4 + 7672t^3 - 5040t^2)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(7)$$

where $t = (x - x_n)/h$. Evaluating (6) at $t = 1(1)6$ gives a block scheme of the form:

$$A^{(0)}\mathbf{Y}_m = \mathbf{E}\mathbf{y}_n + hd\mathbf{f}(\mathbf{y}_n) + hb\mathbf{F}(\mathbf{Y}_m) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

where

$$\mathbf{Y}_m = \left[y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+6}\right]^T, \mathbf{y}_n = \left[y_{n-5}, y_{n-4}, y_{n-3}, y_{n-2}, y_{n-1}, y_n\right]^T,$$

$$\mathbf{F}(\mathbf{Y}_m) = \left[f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}\right]^T, \mathbf{f}(\mathbf{y}_n) = \left[f_{n-5}, f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n\right]^T,$$

$$A^{(0)} = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \end{bmatrix}, \quad d = \begin{bmatrix} 0&0&0&0&0&\frac{19087}{60480} \\ 0&0&0&0&0&\frac{1139}{3780} \\ 0&0&0&0&0&\frac{137}{448} \\ 0&0&0&0&0&\frac{286}{945} \\ 0&0&0&0&0&\frac{3715}{12096} \\ 0&0&0&0&0&\frac{41}{140} \end{bmatrix}$$

$$b=\begin{bmatrix} \frac{2713}{2520} & \frac{-15487}{20160} & \frac{586}{945} & \frac{-6737}{20160} & \frac{263}{2520} & \frac{-863}{60480} \\ \frac{94}{63} & \frac{11}{1260} & \frac{332}{945} & \frac{-269}{1260} & \frac{22}{315} & \frac{-37}{3780} \\ \frac{81}{56} & \frac{1161}{2240} & \frac{34}{35} & \frac{-729}{2240} & \frac{27}{280} & \frac{-29}{2240} \\ \frac{464}{315} & \frac{128}{315} & \frac{1504}{945} & \frac{58}{315} & \frac{16}{315} & \frac{-8}{945} \\ \frac{725}{504} & \frac{2125}{4032} & \frac{250}{189} & \frac{3875}{4032} & \frac{235}{504} & \frac{-275}{12096} \\ \frac{54}{35} & \frac{27}{140} & \frac{68}{35} & \frac{27}{140} & \frac{54}{35} & \frac{41}{140} \end{bmatrix}$$

The algorithm for implementing the block integrator (8) using matlab is given by,

function output= $f(x,y)$

output=?;

function exactsol= $fr(x)$

exact sol=?;

$x0=?;\ y0=?;\ h=?;$

disp('x-value   Exact Solution   Computed Solution   Error ')

for $j=1:1:4;$

$i=1:6;$

$f0=g(x0,y0);$

$x(i)=x0+i*h;$

$$y(i)=y0+(i*h)*f0+(((i*h)\wedge 2)/2)*dx(x0,y0) +(((i*h)\wedge 3)/6)*ddx(x0,y0)+\ldots$$

$$
\begin{aligned}
& yi(x) = gi_L \\
& mi = toc; \\
& erri = abs(fr(x(i)) - yri); \\
& fprint\, f('\%2.4f\ \ \%3.16f\ \ \%3.16f\ \ \%1.6e\ \%2.4f \\
& \qquad \backslash n',\ x(i),\ \ fr(x(i)),\ yri,\ erri,\ mi) \\
& x0 = x(6);\ y0 = yr(6); \\
& end
\end{aligned}
$$

Note that: $y(x) = g_L$, for $g_L = \alpha_0(x)y_n + h\beta_j(x)f_{n+j},\ j = 0(1)6$

Analysis of Basic Properties of the improved block integrator
Order of the New Block Integrator

Let the linear operator $L\{y(x);h\}$ associated with the block (8) be defined as:

$$L\{y(x);h\} = A^{(0)}Y_m - Ey_n - hdf(y_n) - hbF(Y_m) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Expanding (9) using Taylor series and comparing the coefficients of h gives:

$$L\{y(x);h\} = c_0 y(x) + c_1 hy'(x) + c_2 h^2 y''(x) + \ldots + c_p h^p y^p(x) + c_{p+1} h^{p+1} y^{p+1}(x) + \ldots \ldots\ldots(10)$$

Definition

The linear operator L and the associated continuous linear multistep method (6) are said to be of order p if $c_0 = c_1 = c_2 = \ldots = c_p = 0\ and\ c_{p+1} \neq 0.$ c_{p+1} is called the error constant and the local truncation error is given by:

$$t_{n+k} = c_{p+1} h^{(p+1)} y^{(p+1)}(x_n) + O(h^{p+2}) \ \ldots.(11)$$

For our integrator:

$$L\{y(x);h\}=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\\y_{n+5}\\y_{n+6}\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}\begin{bmatrix}y_n\end{bmatrix}-h\begin{bmatrix}\frac{19087}{60480}&\frac{2713}{2520}&\frac{-15487}{20160}&\frac{586}{945}&\frac{-6737}{20160}&\frac{263}{2520}&\frac{-863}{60480}\\\frac{1139}{3780}&\frac{94}{63}&\frac{11}{1260}&\frac{332}{945}&\frac{-269}{1260}&\frac{22}{315}&\frac{-37}{3780}\\\frac{137}{448}&\frac{81}{56}&\frac{1161}{2240}&\frac{34}{35}&\frac{-729}{2240}&\frac{27}{280}&\frac{-29}{2240}\\\frac{286}{945}&\frac{464}{315}&\frac{128}{315}&\frac{1504}{945}&\frac{58}{315}&\frac{16}{315}&\frac{-8}{945}\\\frac{3715}{12096}&\frac{725}{504}&\frac{2125}{4032}&\frac{250}{189}&\frac{3875}{4032}&\frac{235}{504}&\frac{-275}{12096}\\\frac{41}{140}&\frac{54}{35}&\frac{27}{140}&\frac{68}{35}&\frac{27}{140}&\frac{54}{35}&\frac{41}{140}\end{bmatrix}\begin{bmatrix}f_n\\f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\\f_{n+6}\end{bmatrix}=0 \qquad (12)$$

Expanding (12) in Taylor series gives:

$$\begin{bmatrix}\sum_{j=0}^{\infty}\frac{(h)^j}{j!}y_n^j-y_n-\frac{19087h}{60480}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{2713}{2520}(1)^j-\frac{15487}{20160}(2)^j+\frac{586}{945}(3)^j-\frac{6737}{20160}(4)^j+\frac{263}{2520}(5)^j-\frac{863}{60480}(6)^j\right\}\\\sum_{j=0}^{\infty}\frac{(2h)^j}{j!}y_n^j-y_n-\frac{1139h}{3780}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{94}{63}(1)^j+\frac{11}{1260}(2)^j+\frac{332}{945}(3)^j-\frac{269}{1260}(4)^j+\frac{22}{315}(5)^j-\frac{37}{3780}(6)^j\right\}\\\sum_{j=0}^{\infty}\frac{(3h)^j}{j!}y_n^j-y_n-\frac{137h}{448}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{81}{56}(1)^j+\frac{11}{2240}(2)^j+\frac{34}{35}(3)^j-\frac{729}{2240}(4)^j+\frac{27}{280}(5)^j-\frac{29}{2240}(6)^j\right\}\\\sum_{j=0}^{\infty}\frac{(4h)^j}{j!}y_n^j-y_n-\frac{286h}{945}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{464}{315}(1)^j+\frac{128}{315}(2)^j+\frac{1504}{945}(3)^j+\frac{58}{315}(4)^j+\frac{16}{315}(5)^j-\frac{8}{945}(6)^j\right\}\\\sum_{j=0}^{\infty}\frac{(5h)^j}{j!}y_n^j-y_n-\frac{315h}{12096}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{725}{504}(1)^j+\frac{2125}{4032}(2)^j+\frac{250}{189}(3)^j+\frac{3875}{4032}(4)^j+\frac{235}{504}(5)^j-\frac{275}{12096}(6)^j\right\}\\\sum_{j=0}^{\infty}\frac{(6h)^j}{j!}y_n^j-y_n-\frac{41h}{140}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{54}{35}(1)^j+\frac{27}{140}(2)^j+\frac{68}{35}(3)^j+\frac{27}{140}(4)^j+\frac{54}{35}(5)^j+\frac{41}{140}(6)^j\right\}\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix} \qquad (13)$$

Hence,

$\bar{c}_0=\bar{c}_1=\bar{c}_2=\bar{c}_3=\bar{c}_4=\bar{c}_5=\bar{c}_6=\bar{c}_7=0,\ \bar{c}_8=\left[0.010(-03), 0.006(-03),\ 0.08(-03),\ 0.006(-03),\ 0.009(-03), -0.001(-03)\right]^T$

. Therefore, the new block integrator is of order seven.

Zero Stability

Definition

The block integrator (8) is said to be zero-stable, if the roots $z_s, s = 1, 2, ..., k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z) = \det(z\mathbf{A}^{(0)} - \mathbf{E})$ satisfies $|z_s| \leq 1$ and every root satisfying $|z_s| \leq 1$ have multiplicity not exceeding the order of the differential equation. Moreover, as $h \to 0$, $\rho(z) = z^{r-\mu}(z-1)^{\mu}$ where μ is the order of the differential equation, r is the order of the matrices $\mathbf{A}^{(0)}$ *and* $\mathbf{E}$, see Awoyemi *et al.* (2007) for details.

For our new integrator,

$$\rho(z) = \left| z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

$\rho(z) = z^5(z-1) = 0, \Rightarrow z_1 = z_2 = z_3 = z_4 = z_5 = 0, z_6 = 1$. Hence, the new block integrator is zero-stable.

Consistency

The block integrator (8) is consistent since it has order $p = 7 \geq 1$.

Convergence

The new block integrator is convergent by consequence of Dahlquist theorem below.

Theorem Dahlquist (1956)

The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero-stable.

Region of Absolute Stability

Definition

The block integrator (8) is said to be absolutely stable if for a given h, all the roots z_s of the characteristic polynomial $\pi(z, \bar{h}) = \rho(z) + \bar{h}\sigma(z) = 0$ satisfies $z_s < 1,\ s = 1, 2, ..., n$

where $\bar{h} = \lambda h$ *and* $\lambda = \dfrac{\partial f}{\partial y}$.

We shall adopt the boundary locus method for the region of absolute stability of the block integrator. Substituting the test equation $y' = -\lambda y$ into the block formula gives,

$$\mathbf{A}^{(0)}\mathbf{Y}_m(r) = \mathbf{E}y_n(r) - h\lambda\mathbf{D}y_n(r) - h\lambda\mathbf{B}\mathbf{Y}_m(r) \qquad \text{...(15)}$$

Thus:

$$\bar{h}(r) = -\left(\frac{\mathbf{A}^{(0)}Y_m(r) - \mathbf{E}y_n(r)}{\mathbf{D}y_n(r) + \mathbf{B}Y_m(r)}\right) \qquad \text{..........(16)}$$

Writing (16) in trigonometric ratios gives:

$$\bar{h}(\theta) = -\left(\frac{\mathbf{A}^{(0)}Y_m(\theta) - \mathbf{E}y_n(\theta)}{\mathbf{D}y_n(\theta) + \mathbf{B}Y_m(\theta)}\right) \qquad \text{.........(17)}$$

where $r = e^{i\theta}$. Equation (17) is our characteristic/stability polynomial. Applying (17) to our integrator gives,

$$\bar{h}(\theta) = \frac{(\cos 2\theta)(\cos 3\theta)(\cos 4\theta)(\cos 5\theta)(\cos\theta) - (\cos 2\theta)(\cos 3\theta)(\cos 4\theta)(\cos 5\theta)(\cos 6\theta)(\cos\theta)}{\frac{1}{7}(\cos 2\theta)(\cos 3\theta)(\cos 4\theta)(\cos 5\theta)(\cos\theta) - \frac{1}{7}(\cos 2\theta)(\cos 3\theta)(\cos 4\theta)(\cos 5\theta)(\cos 6\theta)(\cos\theta)} \qquad \text{...........(18)}$$

which gives the stability region shown in fig. 1 below.

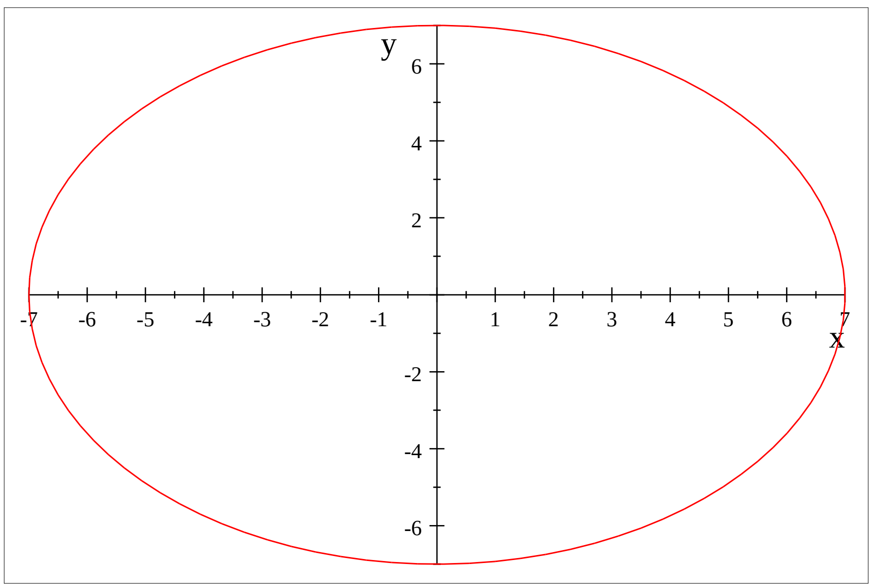

Fig 1: Region of Absolute Stability of the Block Integrator

Numerical Implementations

We shall use the following notation in the tables below;

ERR- |Exact Solution-Computed Result|

ECZ-Error in Chollom and Zirra (2009)

EUY-Error in Umar and Yahaya (2010)

Problem 1

Consider the stiff initial value problem,

$$y' = -y,\ y(0) = 1 \qquad \text{.........................(19)}$$

with the exact solution,

$$y(x) = e^{-x} \qquad \text{...................................(20)}$$

This problem was solved by Umar and Yahaya (2010). They adopted a block method of order four to solve the problem. We compare the result of our new block integrator (8) with their result as shown in table 1.

Problem 2

Consider the non-stiff initial value problem,

$$y' = x + y,\ y(0) = 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

with the exact solution,

$$y(x) = 3e^{x} - x - 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

Chollom and Zirra (2009) solved this problem by apply an L-stable block hybrid Adams method with k=3 and 3 off-grid points. We compare the result of our new block integrator (8) with their result as shown in table 2.

Table 1: Showing the Exact Solution and Computed Result from the New Block Integrator for Problem 1

x	Exact solution	Computed solution	ERR	EUY
0.1000	0.9048374180359595	0.9048374180357143	2.452483e-013	2.5292e-006
0.2000	0.8187307530779818	0.8187307530158730	6.210887e-012	2.0937e-006
0.3000	0.7408182206817178	0.7408182206832924	1.574575e-012	2.0079e-006
0.4000	0.6703200460356393	0.6703200460371952	1.555945e-012	1.6198e-006
0.5000	0.6065306597126334	0.6065306597104390	9.175628e-011	3.1608e-006
0.6000	0.5488116360940264	0.5488116360133064	3.903797e-011	2.7294e-006
0.7000	0.4965853037914095	0.4965853037126733	3.532303e-011	2.5457e-006
0.8000	0.4493289641172216	0.4493289641491866	3.196500e-011	2.1713e-006
0.9000	0.4065696597405991	0.4065696597696055	2.900646e-011	3.1008e-006
1.0000	0.3678794411714423	0.3678794411341661	2.702186e-011	2.7182e-006

Table 2: Showing the Exact Solution and Computed Result from the New Block Integrator for Problem 2

x	Exact solution	Computed solution	ERR	ECZ
0.1000	2.2155127542269430	2.2155127542261903	7.527312e-013	7.7743e-009
0.2000	2.4642082744805096	2.4642082744857143	1.947953e-010	7.7912e-009
0.3000	2.7495764227280093	2.7495764227285715	5.049438e-010	9.8483e-003
0.4000	3.0754740929238107	3.0754740929247621	5.101905e-010	2.6199e-002
0.5000	3.4461638121003846	3.4461638121642858	3.076361e-010	2.8954e-002
0.6000	3.8663564011715277	3.8663564018571430	1.338314e-010	3.1999e-002

0.7000	4.3412581224114293	4.3412581223201132	1.479091e-010	1.7235e-003
0.8000	4.8766227854774042	4.8766227854539638	1.638023e-010	9.1553e-003
0.9000	5.4788093334708492	5.4788093336632890	1.867808e-010	3.1950e-003
1.0000	6.1548454853771375	6.1548454857887007	2.479588e-010	2.2621e-003

Conclusion

In this paper, we have proposed an improved continuous block numerical integrator for the solution of first-order ordinary differential equations. The block integrator proposed was found to be zero-stable, consistent and convergent. The integrator was also found to be computationally reliable on the set of problems it was designed for.

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