The Golden Gravitational Field of a Static Homogeneous Oblate Spheroidal Massive Body

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ABSTRACT

It is well known that Newton's dynamical gravitational field equation and Einstein's geometrical gravitational field equation were derived based upon the Euclidean geometry under the assumption that massive bodies are perfectly spherical in nature. In this paper, we used a golden metric tensor for all gravitational fields in nature to develop Riemannian Laplacian field equation which is assumed to be more general than the Euclidean geometry in order to derive Riemann's dynamical gravitational field equation for a static homogeneous oblate spheroidal massive bodies due correction terms of all order c^{-2} that is not found in Newton's gravitational field equation. Both the Newton's gravitation field equation of motion and Riemann's gravitational field equation are for mathematical analysis and hence physical interpretation and experimental investigation for all bodies in the universe.

KEYWORDS: Golden metric tensor, Riemannian Laplacian Equation, Spherical polar coordinates, Oblate spheroidal coordinates.

Introduction

In the last six decades, the theoretical study of gravitational fields in Newton's dynamical gravitational field equation was treated under the assumption that massive bodies are perfectly spherical in nature (Weinberg, 1972, Anderson, 1967, Bergmann, 1987 Lumbi, Howusu, 2014). In the same way, Einstein's Geometrical gravitational field equation was treated under the assumption that the sun is exclusively a perfect sphere as seen in Schwartz child's solution to Einstein's Geometrical gravitational field equation (Weinberg, 1972, Anderson, 1967, Bergmann, 1987, Howusu, 2013). It is well known that Newton's dynamical gravitational field equation and Einstein's geometrical gravitational field equation were derived based upon the Euclidean Laplacian operator (Howusu, 2013, Lumbi, Howusu, Liman, 2014, Lumbi, Howusu, 2014, Lumbi, Howusu, Tsaku, Nwagbara, 2014, Howusu, 2010). Recently, research has shown that all rotating bodies example the planets, sun, stars and the Galaxies in the universe are either oblate spheroidal or prolate spheroidal in shape (Lumbi, Howusu, 2014, Lumbi, Howusu, Tsaku, Nwagbara, 2014).

Theoretical Analysis

Consider a static homogeneous oblate spheroidal massive body of rest mass M and radius η_0. The Riemannian dynamical gravitational field equation is given explicitly by (Lumbi, Howusu, Liman, 2014, Lumbi, Howusu, Tsaku, Nwagbara, 2014, Howusu, 2010, Howusu, 2009)

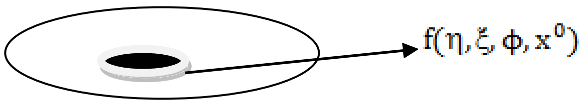

Figure 1.

Density ρ_0

Radius η_0

Mass M

$$\nabla_R^2 f(\eta, \xi, \phi, x^0) = 4\pi G \rho_0 (\eta, \xi, \phi, x^0) \qquad (1)$$

Where ∇_R^2 is Riemannian Laplacian

Given

$$\nabla_R^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \left\{ \sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right\} \qquad (2)$$

and g_{uv} is the metric tensor (either schwarztchild or great metric tensor or golden metric tensor). Here we use golden metric tensor of oblate spheroidal coordinates.

In spherical polar coordinates, Golden metric tensor is given by:

$$g_{11} = \left\{ 1 + \frac{2}{c^2} f(r, \theta, \phi, x^0) \right\}^{-1} \qquad (3)$$

$$g_{22} = r^2 \left\{ 1 + \frac{2}{c^2} f(r, \theta, \phi, x^0) \right\}^{-1} \qquad (4)$$

$$g_{33} = r^2 \sin^2\theta \left\{ 1 + \frac{2}{c^2} f(r, \theta, \phi, x^0) \right\}^{-1} \qquad (5)$$

$$g_{00} = -\left\{1 + \frac{2}{c^2} f(r, \theta, \phi, x^0)\right\} \tag{6}$$

$$g_{uv} = 0, \text{otherwise} \tag{7}$$

To express Golden metric tensor in oblate spheroidal coordinates, we need to transform as we shall see later.
$(r, \theta, \phi, x^0) \rightarrow (\eta, \xi, \phi, x^0)$

Note that:

$$g_{11} = h_1^2 \left(1 + \frac{2}{c^2} f\right)^{-1} \tag{8}$$

$$g_{22} = h_2^2 \left(1 + \frac{2}{c^2} f\right)^{-1} \tag{9}$$

$$g_{33} = h_3^2 \left(1 + \frac{2}{c^2} f\right)^{-1} \tag{10}$$

$$g_{00} = -\left(1 + \frac{2}{c^2} f\right) \tag{11}$$

$$g_{uv} = 0, \text{otherwise} \tag{12}$$

Where h_1, h_2, h_3 are the scale factors in oblate spheroidal coordinates to be evaluated.
Now consider Riemannian Laplacian

$$\nabla_R^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} \left\{\sqrt{g} g^{1\beta} \frac{\partial}{\partial x^\beta}\right\} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^2} \left\{\sqrt{g} g^{2\beta} \frac{\partial}{\partial x^\beta}\right\}$$

$$+ \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^3} \left\{\sqrt{g} g^{3\beta} \frac{\partial}{\partial x^\beta}\right\} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^0} \left\{\sqrt{g} g^{0\beta} \frac{\partial}{\partial x^\beta}\right\} \tag{13}$$

$$\nabla_R^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} \left\{\sqrt{g} g^{11} \frac{\partial}{\partial x^1}\right\} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} \left\{\sqrt{g} g^{12} \frac{\partial}{\partial x^2}\right\}$$

$$+ \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} \left\{\sqrt{g} g^{13} \frac{\partial}{\partial x^3}\right\} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^1} \left\{\sqrt{g} g^{10} \frac{\partial}{\partial x^0}\right\}$$

$$+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{21}\frac{\partial}{\partial x^1}\right\}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{22}\frac{\partial}{\partial x^2}\right\}$$

$$+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{23}\frac{\partial}{\partial x^3}\right\}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{20}\frac{\partial}{\partial x^0}\right\}$$

$$+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{31}\frac{\partial}{\partial x^1}\right\}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{32}\frac{\partial}{\partial x^2}\right\}$$

$$+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{33}\frac{\partial}{\partial x^3}\right\}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{30}\frac{\partial}{\partial x^0}\right\}$$

$$+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{01}\frac{\partial}{\partial x^1}\right\}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{02}\frac{\partial}{\partial x^2}\right\}$$

$$+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{03}\frac{\partial}{\partial x^3}\right\}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{00}\frac{\partial}{\partial x^0}\right\}$$

$$= 4\pi G\rho_0(r,\theta,\phi,x^0) \qquad (14)$$

The above Riemannian field equation reduced to:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^1}\left\{\sqrt{g}g^{11}\frac{\partial}{\partial x^1}\right\}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}\left\{\sqrt{g}g^{22}\frac{\partial}{\partial x^2}\right\}$$

$$+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}\left\{\sqrt{g}g^{33}\frac{\partial}{\partial x^3}\right\}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^0}\left\{\sqrt{g}g^{00}\frac{\partial}{\partial x^0}\right\}$$

$$= 4\pi G\rho_0(r,\theta,\phi,x^0) \qquad (15)$$

By definition g is the determinant of the Golden metric tensor

$$g = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 \\ 0 & 0 & g_{33} & 0 \\ 0 & 0 & 0 & g_{00} \end{pmatrix} \qquad (16)$$

$$g = g_{11} \cdot g_{22} \cdot g_{33} \cdot g_{00} \tag{17}$$

$$= -h_1^2\ h_2^2\ h_3^2 \left(1 + \frac{2}{c^2} f\right)^{-2} \left(1 + \frac{2}{c^2} f\right)^{1} = -h_1^2\ h_2^2\ h_3^2 \left(1 + \frac{2}{c^2} f\right)^{-2}$$

$$\sqrt{g} = h_1\ h_2\ h_3 \left(1 + \frac{2}{c^2} f\right)^{-1} \tag{18}$$

For oblate spheroidal coordinate

$$g^{11} = \frac{1}{g_{11}} = \frac{1}{h_1^2 \left(1 + \frac{2}{c^2} f\right)^{-1}} = \frac{\left(1 + \frac{2}{c^2} f\right)}{h_1^2} \tag{19}$$

$$g^{22} = \frac{1}{g_{22}} = \frac{1}{h_2^2 \left(1 + \frac{2}{c^2} f\right)^{-1}} = \frac{\left(1 + \frac{2}{c^2} f\right)}{h_2^2} \tag{20}$$

$$g^{33} = \frac{1}{g_{33}} = \frac{1}{h_3^2 \left(1 + \frac{2}{c^2} f\right)^{-1}} = \frac{\left(1 + \frac{2}{c^2} f\right)}{h_3^2} \tag{21}$$

$$g^{00} = \frac{1}{g_{00}} = -\frac{1}{\left(1 + \frac{2}{c^2} f\right)} = -\left(1 + \frac{2}{c^2} f\right) \tag{22}$$

$$g^{uv} = 0, \quad \text{otherwise} \tag{23}$$

Note also the scale factors for oblate spheroidal coordinates are:

$$h_1 = a \left(\frac{\eta^2 + \xi^2}{1 + \eta^2}\right)^{1/2} \tag{24}$$

$$h_2 = a \left(\frac{\eta^2 + \xi^2}{1 + \xi^2}\right)^{1/2} \tag{25}$$

$$h_3 = a[(1 + \eta^2)(1 + \xi^2)]^{1/2} \tag{26}$$

By making use of equation (1.27) above and substituting h_1, h_2 and h_3 . We get:

$$\sqrt{g} = a\left(\frac{\eta^2+\xi^2}{1+\eta^2}\right)^{1/2} \cdot a\left(\frac{\eta^2+\xi^2}{1+\xi^2}\right)^{1/2} \cdot a[(1+\eta^2)(1+\xi^2)]^{1/2} \times \left(1+\frac{2}{c^2}f\right)^{-1}$$

$$= [a^6(\eta^2+\xi^2)^2]^{1/2}\left(1+\frac{2}{c^2}f\right)^{-1} \qquad (27)$$

$$\sqrt{g} = a^3(\eta^2+\xi^2)\left(1+\frac{2}{c^2}f\right)^{-1} \qquad (28)$$

Putting $\sqrt{g}$, g^{11}, g^{22}, g^{33} and g^{00} into the field equation we get

$$\frac{1}{a^3(\eta^2+\xi^2)\left(1+\frac{2}{c^2}f\right)^{-1}}\frac{\partial}{\partial\eta}\left\{a^3(\eta^2+\xi^2)\left(1+\frac{2}{c^2}f\right)^{-1}\frac{\left(1+\frac{2}{c^2}f\right)}{\frac{a^2(\eta^2+\xi^2)}{(1+\eta^2)}}\frac{\partial}{\partial\eta}f\right\}$$

$$+\frac{1}{a^3(\eta^2+\xi^2)\left(1+\frac{2}{c^2}f\right)^{-1}}\frac{\partial}{\partial\xi}\left\{a^3(\eta^2+\xi^2)\left(1+\frac{2}{c^2}f\right)^{-1}\frac{\left(1+\frac{2}{c^2}f\right)}{\frac{a^2(\eta^2+\xi^2)}{(1+\xi^2)}}\frac{\partial}{\partial\xi}f\right\}$$

$$+\frac{1}{a^3(\eta^2+\xi^2)(1+\frac{2}{c^2}f)^{-1}}\frac{\partial}{\partial\phi}\left\{a^3(\eta^2+\xi^2)\left(1+\frac{2}{c^2}f\right)^{-1}\frac{\left(1+\frac{2}{c^2}f\right)}{a^2(1+\eta^2)(1+\xi^2)}\frac{\partial}{\partial\phi}f\right\}$$

$$-\frac{1}{c^2(1+\frac{2}{c^2}f)^{-1}}\frac{\partial}{\partial t}\left\{\left(1+\frac{2}{c^2}f\right)^{-1}\left(1+\frac{2}{c^2}f\right)^{-1}\frac{\partial}{\partial t}f\right\}$$

$$= 4\pi G\rho(\eta,\xi,\phi,x^0) \qquad (29)$$

$$\frac{\left(1+\frac{2}{c^2}f\right)}{(\eta^2+\xi^2)}\frac{\partial}{\partial\eta}\left\{\frac{(\eta^2+\xi^2)(1+\eta^2)}{a^2(n^2+\xi^2)}\frac{\partial}{\partial\eta}f\right\} + \frac{\left(1+\frac{2}{c^2}f\right)}{(\eta^2+\xi^2)}\frac{\partial}{\partial\xi}\left\{\frac{(\eta^2+\xi^2)(1+\xi^2)}{a^2(n^2+\xi^2)}\frac{\partial}{\partial\xi}f\right\}$$

$$+(1+\frac{2}{c^2}f)\frac{\partial}{\partial\phi}\left\{\frac{1}{a^2(1-\eta^2)(1+\xi^2)}\frac{\partial}{\partial\phi}f\right\} - \frac{\left(1+\frac{2}{c^2}f\right)}{c^2}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}f\right\}$$

$$= 4\pi G\rho(\eta, \xi, \phi, x^0) \quad (30)$$

$$\left(1 + \frac{2}{c^2}f\right)\left[\frac{1}{a^2(\eta^2+\xi^2)}\frac{\partial}{\partial\eta}\left\{(1+\eta^2)\frac{\partial}{\partial\eta}f\right\} + \frac{1}{a^2(\eta^2+\xi^2)}\frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f\right\} + \frac{1}{a^2(1-\eta^2)(1+\xi^2)}\frac{\partial^2}{\partial\phi^2}f - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}f\right]$$

$$= 4\pi G\rho_0(\eta, \xi, \phi, x^0) \quad (31)$$

Static homogeneous oblate spheroidal distribution of massive body
Note: (1) No Time variation (static)

(2) No azimuthal angle ϕ (symmetry)

$$\left(1 + \frac{2}{c^2}f\right)\left[\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f\right\} + \frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f\right\}\right] = 4\pi G\rho_0(\eta, \xi)\, a^2(\eta^2+\xi^2) \quad (32)$$

$$\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f\right\} + \frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f\right\} + \frac{2}{c^2}f\left[\frac{\partial}{\partial\eta}\{(1-\eta^2)\}\frac{\partial}{\partial\eta}f\right] + \frac{2}{c^2}f\left[\frac{\partial}{\partial\xi}\{1+\xi^2\}\frac{\partial}{\partial\xi}f\right]$$

$$= a^2(\eta^2+\xi^2)\cdot 4\pi G\rho_0(\eta, \xi) \quad (33)$$

Now equation (33) can be break into three parts:

First part
(α) exterior homogeneous equation

$$\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f\right\} + \frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f\right\} + \frac{2}{c^2}f\left[\frac{\partial}{\partial\xi}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f\right\}\right] + \frac{2}{c^2}f\left[\frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{2}{\partial\xi}f\right\}\right]$$

$$= 0, \qquad \xi > \xi_0 \quad (34)$$

Second Part
(β) Interior homogeneous equation

$$\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f\right\} + \frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f\right\} + \frac{2}{c^2}f\left[\frac{\partial}{\partial\xi}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}f\right\}\right] + \frac{2}{c^2}f\left[\frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f\right\}\right]$$

$$= a^2(\xi^2+\eta^2)4\pi G\rho_0(\eta, \xi), \qquad \xi < \xi_0 \quad (35)$$

Third Part

(γ) Interior non homogeneous equation

$$\frac{\partial}{\partial\eta}\left\{(1+\eta^2)\frac{\partial}{\partial\eta}f\right\}+\frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f\right\}+\frac{2}{c^2}f\left[\frac{\partial}{\partial\eta}\left\{(1+\eta^2)\frac{\partial}{\partial\eta}f\right\}\right]+\frac{2}{c^2}\left[\frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial}{\partial\xi}f\right\}\right]$$

$$= a^2(\xi^2+\eta^2)4\pi G\rho_0(\eta,\xi), \qquad \xi < \xi_0 \qquad (36)$$

Note that:

- The first two terms of equation (35) are Newton's gravitational field equations for the gravitational scalar potential f due to a distribution of mass density $\rho_0(\eta,\xi)$

- While the last two terms are the added terms or contribution due to Riemann and is referred to as Riemann gravitational field equation need to be solved.

Results and Discussion

In this research we showed how to formulate the Riemannian Gravitational Field Equation for static homogeneous oblate spheroidal massive bodies using the Riemannian Laplacian operator. The Riemannian Gravitational Field Equation for static homogeneous oblate spheroidal massive bodies is given by equation (35). The immediate consequences of the result obtained in this paper are:-

i. It can be solved by seeking the exterior and interior solution to obtain the Riemannian dynamical gravitational scalar potential exterior and interior to the body for static homogeneous oblate spheroidal massive bodies.

ii. The Riemannian dynamical gravitational scalar potential exterior and interior can be applied to the motion of test particles in the gravitational fields under study.

iii. The Riemannian dynamical gravitational scalar potential exterior and interior to the body can be substituted into any dynamical gravitational equation of motion (Newton's and Einstein's) for static homogeneous oblate spheroidal massive bodies to obtain corresponding revisions to the planetary equations of motion and hence to the planetary parameters. This research can be applied to investigate the motion of an earth satellite and particles whose geometrical shapes are oblate spheroidal.

Conclusion

In this research paper we showed how to formulate the Riemannian Gravitational Field Equation for static homogeneous oblate spheroidal massive bodies using the Riemannian Laplacian operator. A profound philosophical inference is suggested by merely looking at this work, it shows that Riemann's gravitational field equations for a spheroidal body could also be linear and separable, and hence can be solve in terms of the well known special functions of mathematical physics, the Legendre function.

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