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Convergence of Some Two-Step Random Iterative Procedures

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Abstract

In this article, we prove some theorems to approximate fixed point of S. Reich operator on some random two-step iterative procedure, namely: Random S. Random Thianwan and Random Picard-Mann iterative procedures. The results obtained at the end of the research show the convergence of some random iterative schemes in fixed point theory.

Keywords: Random operator, contractive condition, Random S-Iterative procedure, Random Thianwan Iterative Procedure and Random Picard-Mann Iterative procedure.

Introduction

The idea of random fixed point theorems for random contraction mappings on separable complete metric spaces was introduced by Spacek (1955) and Han (1961), since then there have been several studies in this area. Akewe and Okeke (2015) use the general class of contractive-like operators introduced by Bosede and Rhoades (J. Adv. Math. Stud. 3(2):1-3, 2010) to prove strong convergence and stability results for Picard-Mann hybrid iterative schemes considered in a real normed linear space. Alber and Guerre-Delabriere (1997) introduced the notion of weakly contractive mappings on Hilbert spaces. It lies between those which satisfy Banach's contraction principle and contractive maps. Agawal and Gupta (2016) used Kannan type and Chatterjea contractions to obtain some random fixed point results for multi-valued contractive conditions in complete metric spaces. For more work on random fixed point theorems and random operators, see [Khan (2001), Achari (1983), Agrawal and Gupta (2016), Kravvaritis and Papageorgio (1989) and Rashwan (2011)].

The purpose of this paper is to establish some approximations to fixed point regarding S. Reich operator on some random two-step iterative procedure, namely: Random S., Random Thianwan and Random Picard-Mann iterative procedures.

Preliminaries and Motivation

In this section, we revisit some basic definitions and overview of the fundamental results.

Definition 2.1 [Beg and Abbas (2011)**]:** A random operator $T(\omega)$ on a space X is said to be:

a) $T(\omega)[\alpha x_1 + \beta x_2] = \alpha T(\omega)x_1 + \beta T(\omega)x_2$ almost surely for all $x_1, x_2 \in D(T),\ \alpha, \beta$ scalar;
b) bounded if there exists a nonnegative real-valued random variable $M(\omega)$ such that for all $x \in D(T)$,
$||T(\omega)x|| \leq M(\omega)||x||$ almost surely.
c) Continuous at x_0 if $\lim_{n\to\infty}||x_n - x_0|| = 0$ implies
$$\lim_{n\to\infty}||T(\omega)x_n - T(\omega)x_0|| = 0 \quad \text{almost surely}$$
d) Stochastically continuous if for every $x \in D(T)$ and every $\epsilon > 0$,
$$\lim_{||y-x||\to 0} \mu(\{\omega : ||T(\omega)y - T(\omega)x|| > \epsilon\}) \to 0.$$

For any given mapping $T : X \to Y$, every solution x of the equation

$Tx = x$ 2.1
is called a fixed point.

Definition 2.2 [Hans (1961)]**:** Let X and Z be two separable Banach spaces, T a mapping of the space X into space Z and z a fixed element of Z. If Σ denotes the set of those element $x \in X$ for which the equality $Tx = z$ holds. That is $\Sigma = \{x: T(w) = z\}$, then any $x \in \Sigma$ will be called a solution of the operator equation.

Theorem 2.1 [Hans (1961)]**:** Let T be an almost surely continuous random transformation of Cartesian product space Ω x X into the space X satisfying the condition:

$$\mu(\left\{w: ||T^n(w,x) - T^n(w,y)|| \leq (1 - \frac{1}{n}||x - y||\right\} = 1 \qquad 2.2$$

where $\omega \in \Omega, x \in X$ and $n = 1, 2, \ldots,$ we set $T^1(\omega, x) = T(\omega, x)$ and $T^{n+1}(\omega, x) = T[\omega, T^n(\omega, x)]$. Then there exist a generalized random variable $\emptyset$ with values in the space X satisfying the relation:
$\mu\{\omega: T[\omega, \emptyset(\omega)] = \emptyset(\omega)\}$ and $\mu\{\omega: ||T(\omega, x) - T(\omega, y)|| \leq c(\omega)||x - y||\} = 1$
for all $x, y \in X$.

Definition 2.3 [Beg and Abbas (2011)**]:** A mapping $T: \Omega \times X \rightarrow X$ is called a random operator if for each $x \in X, T(.,x)$ is measurable.

Definition 2.4 [Beg and Abbas (2011)**]:** A measurable mapping $T: \Omega \rightarrow X$ is random fixed point of the random operator $T: \Omega \times X \rightarrow X$ if and only if $T(\omega, x(\omega)) = x(\omega)$ for each $\omega \in \Omega$
We denote the set of random fixed points of T by$RF(T)$.
If the map $T: \Omega \times X \rightarrow X$ has a random fixed point then for each $\omega \in \Omega, T(\omega,.)$ has a fixed point in T. However, the converse is not true.

Lemma 2.1 [Reich (1971)]**:** Prove that if their exist nonnegative numbers a, b, c satisfying $a + b + c < 1$ such that, for each $x, y \in X$, then

$$d(T(x), T(y)) < ad(x, T(x)) + bd(y, T(y)) + cd(x, y). \qquad 2.3$$

Lemma 2.2 [Berinde (2007)]**:** Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be a sequence of nonnegative numbers and $0 \leq q < 1$ so that

$$a_{n+1} \leq qa_n + b_n, \text{for all } n \geq 0$$

i. If $\lim_{n\to\infty} b_n = 0$ then $\lim_{n\to\infty} a_n = 0$.
ii. If $\sum_{n=0}^{\infty} a_n < \infty$ then $\sum_{n=0}^{\infty} b_n < \infty$

If $q = 1$, then the above result holds in a weaker form.

Definition 2.5 (Random Mann Iteration process) **[**Beg and Abbas (2011)**]:** Let $T: \Omega \times X \rightarrow X$ be a random operator where X is a nonempty convex subset of a separable Banach space X. The random Mann Iteration process is a sequence of mappings $\{x_n\}$ de fined by:

$$x_{n+1}(\omega) = (1 - \alpha_n)x_n(\omega) + \alpha_n T(\omega, x_n(\omega)) \qquad 2.4$$

for all $\omega \in \Omega$, $n \geq 0$ where $0 \leq \alpha_n \leq 1$ and $x_0: \Omega \rightarrow X$ is an arbitrary measurable mapping.

Definition 2.6 (Random S-Iteration process) **[**Beg and Abbas (2011)**]:** Let $T: \Omega \times X \rightarrow X$ be a random operator where X is a nonempty convex subset of a separable Banach space X. The random Mann Iteration process is a sequence of mappings $\{x_n\}$ de fined by:

$$x_{n+1}(\omega) = (1 - \alpha_n)T(\omega, y_n(\omega)) + \alpha_n T(\omega, y_n(\omega))$$
$$y_n(\omega) = (1 - \beta_n)x_n(\omega) + \beta_n T(\omega, x_n(\omega)) \qquad 2.5$$

for all $\omega \in \Omega, n \geq 0$ where $0 \leq \alpha_n, \beta_n \leq 1$ and $x_0: \Omega \rightarrow X$ is an arbitrary measurable mapping.

Definition 2.7 (Random Thianwan Iteration process) **[**Beg and Abbas (2011)**]:** Let $T: \Omega \times X \rightarrow X$ be a random operator where X is a nonempty convex subset of a separable Banach space X. The random Mann Iteration process is a sequence of mappings $\{x_n\}$ defined by:

$x_{n+1}(\omega) = (1-\alpha_n)y_n(\omega) + \alpha_n T(\omega, y_n(\omega))$
$y_n(\omega) = (1-\beta_n)x_n(\omega) + \beta_n T(\omega, x_n(\omega))$ 2.6
for all $\omega \in \Omega$, $n \geq 0$ where $0 \leq \alpha_n, \beta_n \leq 1$ and $x_0: \Omega \to X$ is an arbitrary measurable mapping.

Definition 2.8 (Random Picard-Mann Iteration process) **[**Beg and Abbas (2011)**]:** Let T: $\Omega \times X \to X$ be a random operator where X is a nonempty convex subset of a separable Banach space X. The random Mann Iteration process is a sequence of mappings $\{x_n\}$ de fined by:
$x_{n+1}(\omega) = T(\omega, y_n(\omega))$
$y_n(\omega) = (1-\alpha_n)x_n(\omega) + \alpha_n T(\omega, x_n(\omega))$ 2.7
for all $\omega \in \Omega$, $n \geq 0$ where $0 \leq \alpha_n, \beta_n \leq 1$ and $x_0 : \Omega \to X$ is an arbitrary measurable mapping

Results and Discussion

In this section, the main results in this article are discussed.

Theorem 3.1: Let $(X, ||.||)$ be a separable normed linear space and $T : X \to X$ be a self map with a fixed point $x(\omega)$ satisfying (2.3) for which $x, y \in X$, $0 \leq a+b+c < 1$, $\omega \in \Omega$. Let $\{x_n\}_{n=0}^{\infty}$ be the Random S-Iterative scheme defined by (2.5), where $\{a_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequence in [0,1]. Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $x(\omega)$.

Proof. Using Lemma 2.2 with equation 2.3, we have

$$\left\|x_{n+1}(\omega) - x(\omega)\right\| = \left\|(1-\alpha_n)T(\omega, y_n(\omega)) + \alpha_n T(\omega, y_n(\omega)) - x(\omega)\right\| \qquad 3.1$$

$$\leq \left\|(1-\alpha_n)T(\omega, y_n(\omega)) - (1-\alpha_n)x(\omega)\right\| + \left\|\alpha_n T(\omega, y_n(\omega)) - \alpha_n x(\omega)\right\|$$

$$= (1-\alpha_n)\left\|T(\omega, y_n(\omega)) - x(\omega)\right\| + \alpha_n\left\|T(\omega, y_n(\omega)) - x(\omega)\right\|$$

$$\leq (1-\alpha_n)\left[a\left\|x(\omega) - y_n(\omega)\right\| + b\left\|x(\omega) - T(\omega, x(\omega))\right\| + c\left\|y_n(\omega) - T(\omega, y_n(\omega))\right\|\right]$$
$$+ \alpha_n\left[a\left\|x(\omega) - y_n(\omega)\right\| + b\left\|x(\omega) - T(\omega, x(\omega))\right\|\right.$$
$$\left. + c\left\|y_n(\omega) - T(\omega, y_n(\omega))\right\|\right]$$

$$\leq a(1-\alpha_n)\left\|x(\omega) - y_n(\omega)\right\| + a\alpha_n\left\|x(\omega) - y_n(\omega)\right\| \qquad 3.2$$

$$\leq a(1-\alpha_n)\left[\left\|(1-\beta_n)x_n(\omega) + \beta_n T(\omega, x_n(\omega)) - x(\omega)\right\|\right]$$
$$+ a\alpha_n\left[\left\|(1-\beta_n)x_n(\omega) + \beta_n T(\omega, x_n(\omega)) - x(\omega)\right\|\right]$$

$$\leq a(1-\alpha_n)\left[(1-\beta_n)\left\|x_n(\omega) - x(\omega)\right\| + \beta_n\left\|T(\omega, x_n(\omega)) - T(\omega, x(\omega))\right\|\right]$$
$$+ a\alpha_n\left[\left\|x_n(\omega) - x(\omega)\right\| + \beta_n\left\|T(\omega, x_n(\omega)) - T(\omega, x(\omega))\right\|\right]$$

$$\leq a(1-\alpha_n)(1-\beta_n)\left\|x_n(\omega) - x(\omega)\right\|$$
$$+ a(1-\alpha_n)\beta_n\left[a\left\|x(\omega) - y_n(\omega)\right\| + b\left\|x(\omega) - T(\omega, x(\omega))\right\|\right.$$
$$\left. + c\left\|x_n(\omega) - T(\omega, x_n(\omega))\right\|\right]$$
$$+ a\beta_n\left[a\left\|x(\omega) - y_n(\omega)\right\| + b\left\|x(\omega) - T(\omega, x(\omega))\right\|\right.$$
$$\left. + c\left\|x_n(\omega) - T(\omega, x_n(\omega))\right\|\right]$$

$$\leq a(1-\alpha_n)(1-\beta_n)\left\|x_n(\omega) - x(\omega)\right\| + (a\beta_n - a\beta_n\alpha_n)a\left\|x_n(\omega) - x(\omega)\right\|$$
$$+ a\alpha_n(1-\beta_n)\left\|x_n(\omega) - x(\omega)\right\| + a^2\beta_n\left\|x_n(\omega) - x(\omega)\right\|$$

$$\leq [a(1-\alpha_n)(1-\beta_n) + (a\beta_n - a\beta_n\alpha_n)a + a^2\beta_n]\left\|x_n(\omega) - x(\omega)\right\| \qquad 3.3$$

Applying lemma 2.2 and the fact that $0 \leq a+b+c<1$, *then*

$$\left\|x_{n+1}(\omega)-x(\omega)\right\| \leq \left\|x_n(\omega)-x(\omega)\right\| \qquad 3.4$$

hence $\lim_{n\to\infty}\left\|x_n(\omega)-x(\omega)\right\| = 0$

which implies $x_n(\omega) \to x(\omega)$ *as* $n \to \infty$ *for all* $\omega \in \Omega$.

The sequence $x_n(\omega)$ *converges strongly to the fixed point* $x(\omega)$. ■

Theorem 3.2: Let $(X, \|.\|)$ be a separable normed linear space and $T : X \to X$ be a self map with a fixed point $x(\omega)$ satisfying (2.3) for which $x, y \in X$, $0 \leq a+b+c<1$, $\omega \in \Omega$. Let $\{x_n\}_{n=0}^{\infty}$ be the Random Thianwan Iterative scheme defined by (2.6), where $\{a_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequence in [0,1]. Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $x(\omega)$.

***Proof*:** Consider Lemma 2.2 with equation 2.3, then

$$\left\|x_{n+1}(\omega)-x(\omega)\right\| = \left\|(1-\alpha_n)y_n(\omega)+\alpha_n T\big(\omega, y_n(\omega)\big)-x(\omega)\right\| \qquad 3.5$$

$$\leq \left\|(1-\alpha_n)y_n(\omega)-(1-\alpha_n)x(\omega)\right\| + \left\|\alpha_n T\big(\omega, y_n(\omega)\big)-\alpha_n x(\omega)\right\|$$

$$= (1-\alpha_n)\left\|y_n(\omega)-x(\omega)\right\| + \alpha_n\left\|T\big(\omega, y_n(\omega)\big)-x(\omega)\right\|$$

$$= (1-\alpha_n)\left\|y_n(\omega)-x(\omega)\right\| + \alpha_n\left\|T\big(\omega, y_n(\omega)\big)-T(\omega, x(\omega)\right\|$$

$$\leq (1-\alpha_n)\left\|y_n(\omega)-x(\omega)\right\| + \alpha_n\Big[a\left\|y_n(\omega)-x(\omega)\right\| + b\left\|y_n(\omega)-T\big(\omega, y_n(\omega)\big)\right\| + c\left\|x(\omega)-T\big(\omega, x(\omega)\big)\right\|\Big]$$

$$= (1-\alpha_n)\left\|y_n(\omega)-x(\omega)\right\| + a\alpha_n\left\|y_n(\omega)-x(\omega)\right\|$$

$$= [(1-\alpha_n)+a\alpha_n]\left\|y_n(\omega)-x(\omega)\right\| \qquad 3.6$$

$$\leq [(1-\alpha_n)+a\alpha_n]\left\|(1-\beta_n)x_n(\omega)+\beta_n T\big(\omega, x_n(\omega)\big)-x(\omega)\right\|$$

$$\leq [(1-\alpha_n)+a\alpha_n]\Big[\left\|(1-\beta_n)x_n(\omega)-x(\omega)\right\| + \left\|\beta_n T\big(\omega, x_n(\omega)\big)-x(\omega)\right\|\Big]$$

$$\leq [(1-\alpha_n)+a\alpha_n][(1-\beta_n)\left\|x_n(\omega)-x(\omega)\right\| + \beta_n\left\|T\big(\omega, x_n(\omega)\big)-x(\omega)\right\|]$$

$$\leq [(1-\alpha_n)+a\alpha_n]\Big[(1-\beta_n)\left\|x_n(\omega)-x(\omega)\right\| + \beta_n\Big[a\left\|x_n(\omega)-x(\omega)\right\| + b\left\|x_n(\omega)-T\big(\omega, x_n(\omega)\big)\right\| + c\left\|x(\omega)-T\big(\omega, x(\omega)\big)\right\|\Big]\Big]$$

$$= [(1-\alpha_n)+a\alpha_n]\big[(1-\beta_n)\left\|x_n(\omega)-x(\omega)\right\| + a\beta_n\left\|x_n(\omega)-x(\omega)\right\|\big]$$

$$= [(1-\alpha_n)+a\alpha_n][(1-\beta_n)+a\beta_n]\left\|x_n(\omega)-x(\omega)\right\| \qquad 3.7$$

But $0 \leq a+b+c<1$. *Hence*

$$\left\|x_{n+1}(\omega)-x(\omega)\right\| \leq \left\|x_n(\omega)-x(\omega)\right\| \qquad 3.8$$

$$\lim_{n\to\infty}\left\|x_n(\omega)-x(\omega)\right\| = 0$$

Therefore $x_n(\omega) \to x(\omega)$ *as* $n \to \infty$ *for all* $\omega \in \Omega$.

which means that the random Thianwan iteration process (2.6) converges strongly to the fixed point $x(\omega)$. ■

Theorem 3.3: Let $(X, ||.||)$ be a separable normed linear space and $T : X \to X$ be a self map with a fixed point $x(\omega)$ satisfying (2.3) for which $x, y \in X,\ 0 \le a + b + c < 1,\ \omega \in \Omega$. Let $\{x_n\}_{n=0}^{\infty}$ be the Random Picard-Mann Iterative scheme defined by (2.7), where $\{a_n\}_{n=0}^{\infty}$ is a real sequence in [0,1]. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $x(\omega)$.

***Proof*:** By the same conditions in Theorem 3.2

$$\left||x_{n+1}(\omega) - x(\omega)|\right| = \left||T(\omega, y_n(\omega)) - x(\omega)|\right| \qquad 3.9$$

$$= \left||T(\omega, y_n(\omega)) - T(\omega, x(\omega))|\right|$$

$$\le a\left||y_n(\omega) - x(\omega)|\right| + b\left||y_n(\omega) - T(\omega, y_n(\omega))|\right| + c\left||x(\omega) - T(\omega, x(\omega))|\right|$$

$$\le a\left||y_n(\omega) - x(\omega)|\right| \qquad 3.10$$

$$= a\left||(1-\alpha_n)T(\omega, x_n(\omega)) + \alpha_n T(\omega, x_n(\omega)) - x(\omega)|\right|$$

$$\le a(1-\alpha_n)\left||x_n(\omega) - x(\omega)|\right| + a\alpha_n\left||T(\omega, x_n(\omega)) - x(\omega)|\right|$$

$$= a(1-\alpha_n)\left||x_n(\omega) - x(\omega)|\right| + a\alpha_n\left||T(\omega, x_n(\omega)) - T(\omega, x(\omega))|\right|$$

$$\le a(1-\alpha_n)\left||x_n(\omega) - x(\omega)|\right| + a\alpha_n\left[a\left||x_n(\omega) - x(\omega)|\right| + b\left||x(\omega) - T(\omega, x(\omega))|\right| + c\left||x_n(\omega) - T(\omega, x_n(\omega))|\right|\right]$$

$$= a(1-\alpha_n)\left||x_n(\omega) - x(\omega)|\right| + a^2\alpha_n\left||x_n(\omega) - x(\omega)|\right|$$

$$= [a(1-\alpha_n) + a^2\alpha_n]\left||x_n(\omega) - x(\omega)|\right| \qquad 3.11$$

$$\le \left||x_n(\omega) - x(\omega)|\right| \qquad 3.12$$

Therefore $lim_{n\to\infty}\left||x_n(\omega) - x(\omega)|\right| = 0$. *consequently,* $x_n(\omega) \to x(\omega)$ *as* $n \to \infty$ *for all* $\omega \in \Omega$, *which implies that the random Picard-Mann iteration process (2.7) converges strongly to the fixed point* $x(\omega)$. ■

Conclusion

The results in this article have shown convergence of some random iterative schemes in fixed point theory.

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