

## The Strong Convergence and Stability of Ishikawa and Thianwan Iterative Schemes in a Convex Metric Space

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### ABSTRACT

In this article, we prove some theorems on strong convergence and stability of Thianwan, Ishikawa and Mann iteration for quasi-contractive operator in convex metric spaces. Also, we show the convergence rate of the three iterations through computational results.

**KEYWORDS:** Ishikawa-Iterative Scheme, Thianwan-Iterative Scheme, Convex Metric Space, Strong Convergence, quasi-contractive operator.

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### Introduction

In mathematics, a metric space is a set for which distances between all members of the set are defined. Those distances, taken together, are called a metric on the set. A metric on a space induces topological properties like open and closed sets, lead to the study of more abstract topological spaces.

Many problems in Mathematics and related field can be solved by finding fixed point of a particular operator and algorithms. Finding such points play a prominent role in a number of applications, this is evident in the work of Berinde (2007). In computational mathematics, an iterative method is a mathematical procedure that generates a sequence of improving approximate solutions for a particular class of problems. A specific implementation of an iterative method, including the termination criteria is an algorithm of the iterative method. An iterative method is called convergent if the corresponding sequence converges for any given initial approximations. A mathematically rigorous convergence analysis of an iterative method is usually performed; however, heuristic-based iterative methods are also common. In the problems of finding the root of an equation (or a solution of a system of equations), an iterative method uses an initial guess to generate successive approximations to a solution. In contrast, direct methods attempt to solve the problem by a finite sequence of operations. In the absence of rounding errors, direct methods would deliver an exact solution (like solving a linear system of equations by Gaussian elimination method). Iterative methods are often the only choice for nonlinear equations. However, these methods are often useful even for linear problems involving a large number of variables (sometimes of the order of millions) where direct methods would be prohibitively expensive (and in some cases impossible) even with the best available computing power. The

Picard iteration [Picard (1893)] has also been proved with many contractive conditions in the literature, one of such condition is the Zamfirescu (1972) operator given by:

$$Z_1: d(Tx, Ty) \leq ad(x, y) \quad Z_2: d(Tx, Ty) \leq b [d(x, Tx) + d(y, Ty)] \quad (1.1)$$

$$Z_3: d(Tx, Ty) \leq c [d(x, Ty) + d(y, Tx)]$$

for all  $x, y \in X$  and  $a, b$  and  $c$  are non-negative constants satisfying  $a \in [0, 1), b, c \leq \frac{1}{2}$ . The Theorem is stated as follows:

**THEOREM 1.1:** Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  a mapping for which satisfying (1.1). Then  $T$  has a unique fixed point  $p$  and the Picard iteration  $x_n$  defined by  $x_n = Tx_{n-1}, n \in \mathbb{N}$  converges to  $p$  for any arbitrary but fixed  $x_0 \in X$ .

When the contraction mapping conditions are weaker, then the Picard iteration will no longer converge to a fixed point of the operator, hence, other iteration procedures must be considered. In this case, the mapping under consideration will be called

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for } x, y \in E$$

where  $E$  is a Banach space with a non-expansive operator  $T$ . The non-expansive condition is of particular interest in Banach spaces if  $T$  is assumed to be only non-expansive, that is, it needs not to have a fixed point. Two of the iterations used under the non-expansive conditions in Banach space are Mann and Ishikawa iterations. Mann (1953) defined a more general iteration in a Banach space setting satisfying quasi-nonexpansive operators. The Mann iteration is given as:

$$x_n = (1 - \alpha_n)x_{n-1} + \alpha_nTx_{n-1} \quad (1.2)$$

where  $\alpha_n$  is a sequence of positive numbers in  $[0, 1]$ . Putting  $\alpha_n = 1$  in (1.2) reduces to Picard iteration  $x_n = Tx_{n-1}$ . Liu (1995) introduced the concept of Mann iteration process with errors by the sequence  $\{x_n\}$  defined as follows:

$$x_n = (1 - \alpha_n)x_{n-1} + \alpha_nTx_{n-1} + u_n \quad (1.3)$$

where  $\{x_n\}$  satisfies  $\sum \|u_n\| < \infty$ . This certainly contain (1.1). The proof of the convergence of (1.2) and (1.3) in normed linear setting can be found in Rhoades (1993), Berinde (2004), Rafiq (2006) and Olaleru (2007). Ishikawa (1974), introduced another iteration which is a double Mann iteration and has better convergence rate than Mann iteration. It is given as:

$$x_n = (1 - \alpha_n)x_{n-1} + \alpha_nTy_{n-1}$$

$$y_{n-1} = (1 - \beta_n)x_{n-1} + \beta_nTx_{n-1} \quad (1.4)$$

where  $\alpha_n$  and  $\beta_n$  are sequences of positive numbers in  $[0, 1]$ . The original result of Ishikawa is stated in the following:

**Theorem:** Let  $K$  be a convex compact subset of a Hilbert space  $H, T : K \rightarrow K$  a Lipschitzian pseudocontractive map and  $x_0 \in K$ . The Ishikawa iteration (1.4) satisfying:

- i.  $0 \leq \alpha_n \leq \beta_n \leq 1, n \geq 1$
- ii.  $\lim \beta_n = 0$  as  $n \rightarrow \infty$
- iii.  $\sum \alpha_n \beta_n = 0,$

converges strongly to a fixed point of  $T$ . Convergence and other related results concerning Ishikawa can be found in Hong-Kun (1992), Chidume (1994), Berinde (2004) and Koti *et al.*, (2013).

Another iteration, called Thianwan-iterative process, was introduced by Thianwan (2009) which is independent of Mann and Ishikawa iterations. It is defined in a normed linear space as follows: If  $x_n \in E, E$  a normed space, the sequence  $\{x_n\}$  in  $E$  is defined as:

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n T y_n$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, n \geq 0 \quad (1.5)$$

where  $\alpha_n, \beta_n \subseteq [0, 1]$  with  $\sum \alpha_n = \infty$ .

In this paper, we shall be concerned with the strong convergence and stability results of Thianwan, Mann and Ishikawa iterations in a convex metric space using the quasi-contractive operator defined by Imoru and Olatinwo (2003). We shall also consider the rate of convergence of the three iterations using MATLAB for the numerical computations.

### Material and Methods

In this section, we shall discuss the Thianwan-iterative, Mann and Ishikawa iterations in a convex metric space with quasi-contractive operator.

#### Thianwan and Ishikawa Schemes

Let  $(X, d, W)$  be a convex metric space and  $K$  be a closed subset of  $X$ . Suppose  $T$  is a mapping of  $K$  into itself. For a sequence  $\{x_n\} \in K$ , the Thianwan-iterative scheme is given by:

$$x_n = W(y_{n-1}, T x_{n-1}, \alpha_n)$$

$$y_{n-1} = W(x_{n-1}, T x_{n-1}, \beta_n), n = 1, 2, \dots \quad (2.1)$$

where  $\{\alpha_n\}_{n=1}^{\infty}$  and  $\{\beta_n\}_{n=1}^{\infty}$  are sequences in  $[0, 1]$  with  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Also, for  $\{x_n\} \in K$ , the Ishikawa iteration is defined as:

$$x_n = W(x_{n-1}, T y_{n-1}, \alpha_n)$$

$$y_{n-1} = W(x_{n-1}, T x_{n-1}, \beta_n), n = 1, 2, \dots \quad (2.2)$$

The linear setting of (3.1) and (3.2) are given, respectively, as

$$x_n = \alpha_n y_{n-1} + (1 - \alpha_n) T y_{n-1}$$

$$y_{n-1} = \beta_n x_{n-1} + (1 - \beta_n) T x_{n-1}, n = 1, \quad (2.3)$$

And

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T y_{n-1}$$

$$y_{n-1} = \beta_n x_{n-1} + (1 - \beta_n) T x_{n-1}, n = 1, 2, \dots \quad (2.4)$$

Both iterations (2.1) and (2.2) become Mann iteration when  $\beta_n = 1$ :

$$x_n = W(x_{n-1}, Tx_{n-1}, \alpha_n), n = 1, 2, \dots \tag{2.5}$$

The iteration (2.5) in linear setting is written as:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)Tx_{n-1}, n = 1, 2, \dots \tag{2.6}$$

**Contractive-Type Operators**

Zamfirescu operators are mostly used for contractive-type operators for the study of existence of fixed point. It is stated as follow:

**THEOREM 2.3.1** Let  $X$  be a complete metric space and  $T : X \rightarrow X$  is a Zamfirescu operator satisfying:

$$d(Tx, Ty) \leq h \max \left\{ d(x, y), \frac{1}{2} [d(x, Tx) + d(y, Ty)], \frac{1}{2} [d(x, Ty) + d(y, Tx)] \right\} \tag{2.7}$$

where  $0 \leq h < 1$  and  $x, y \in X$ . For  $x_0 \in X$  and  $x_n \subset X$ , then, the sequence  $\{x_n\}$  converges to a unique fixed point. The contractive condition (2.7) also implies

$$d(Tx, Ty) \leq 2cd(x, Tx) + cd(x, y) \tag{2.8}$$

where  $c = \max \left\{ h, \frac{h}{2-h} \right\}$ .

Osilike (1995) generalized the contractive condition (2.8) as follows: For

$x, y \in X$ , there exists  $a \in [0, 1)$  and  $L \geq 0$  such that

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y) \tag{2.9}$$

Observe that when  $L = 2c$  in (2.9), we have condition (2.8). We shall employ the contractive condition defined by Imoru and Olatinwo (2003) which is more general than (2.7), (2.8) and (2.9). It is given as:

$$d(Tx, Ty) \leq ad(x, y) + \phi(d(x, Tx)) \text{ for } x, y \in X \tag{2.10}$$

where  $a \in [0, 1)$  and  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a monotone increasing function with  $\phi(0) = 0$ .

We shall use the following definition to discuss the concept of Stability in a convex metric space:

**DEFINITION 1:** (Olatinwo, 2011) Let  $(X, d, W)$  be a convex metric space and  $T : X \rightarrow X$  a self-mapping. Suppose that  $F(T) = \{p \in X : Tp = p\}$  is the set of fixed points of  $T$ .

Let  $\{x_n\}_n^\infty = 0 \subset X$  be the sequence generated by an iterative procedure involving  $T$  which is defined by:

$$x_{n+1} = f_{T, \alpha_n}^{x_n}, n \geq 0 \tag{2.11}$$

where  $x_0 \in X$  is the initial approximation and  $f_{T, \alpha_n}^{x_n}$  is some function having convex structure such that  $\alpha_n \in [0, 1]$ . Suppose that  $\{x_n\}$  converges to a fixed point  $p$  of  $T$ . Let  $\{y_n\}_{n=0}^\infty = 0 \subset X$  and set  $\epsilon_n = d y_{n+1}, f_{T, \alpha_n}^{y_n}, n = 0, 1, 2, \dots$ . Then, the iterative procedure (3.11) is said to be  $T$ -stable or stable with respect to  $T$  if and only if  $\lim_{n \rightarrow \infty} \epsilon_n = 0$  implies  $\lim_{n \rightarrow \infty} y_n = p$ .

We shall need the following Lemma in the proof of our main results:

**Lemma 1 (Berinde, 2004)** Let  $\delta$  be a real number such that  $0 \leq \delta < 1$  and  $\{\epsilon_n\}_{n=0}^\infty$  is a sequence of positive numbers such that  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ , then for any sequence of positive numbers  $\{u_n\}_{n=0}^\infty$  satisfying:

$$u_{n+1} \leq \delta u_n + \epsilon_n, n = 0, 1, 2, \dots$$

we have  $\lim_{n \rightarrow \infty} u_n = 0$ .

### Results and Discussion

In this section, we shall discuss the convergence and stability of the schemes (2.1) and (2.2) satisfying the contractive condition (2.10). At the end of computational results, the rate of convergence of Thianwan, Ishikawa and Mann iterations were tabulated.

These are given in the following theorems:

**THEOREM 3.2.1** Let  $K$  be a nonempty closed subset of a convex metric space  $X$  and  $T$  be a self-map of  $K$  satisfying the contractive-type condition (2.10) with  $F(T) \neq \emptyset$ . For  $x_0 \in K$  and  $\{x_n\}$  is a sequence defined by (2.2), then

- (i) the mapping  $T$  satisfying (2.10) has a unique fixed point.
- (ii) the sequence defined by (2.1) converges strongly to the fixed point  $p \in F(T)$ .

### PROOF

(i) Suppose  $p_1, p_2 \in F(T)$  and  $p_1 \neq p_2$ , then  
 $d(p_1, p_1) = d(Tp_1, Tp_2) \leq ad(p_1, p_2) + \varphi(d(Tp_1, p_2))$   
 This implies that,  $d(p_1, p_2) \leq ad(p_1, p_2)$  or  $(1 - a)d(p_1, p_2) \leq 0$   
 Since  $(1 - a)$  is positive and  $0 < (1 - a) < 1$ , then  $d(p_1, p_2) = 0$  by implication  $p_1 = p_2$ . Therefore,  $T$  has a unique fixed point.

(ii) Suppose  $x_0 \in K$  and  $p \in F(T)$ , by (2.1) and (2.10) we have

$$\begin{aligned} d(x_n, p) &= d(W(y_{n-1}, Ty_{n-1}, \alpha_n), p) \\ &\leq \alpha_n d(y_{n-1}, p) + (1 - \alpha_n) d(Ty_{n-1}, p) \\ &\leq \alpha_n d(y_{n-1}, p) + (1 - \alpha_n) [ad(y_{n-1}, p) + \varphi(d(Tp, p))] \\ &= [\alpha_n + (1 - \alpha_n)a] d(y_{n-1}, p) \end{aligned} \tag{3.1}$$

Also, from (2.1)

$$\begin{aligned}
 d(y_{n-1}, p) &= d(W(x_{n-1}, Tx_{n-1}, \beta_n), p) \leq \beta_n d(x_{n-1}, p) + (1 - \beta_n) d(Tx_{n-1}, p) \\
 &\leq \beta_n d(x_{n-1}, p) + (1 - \beta_n) [ad(x_{n-1}, p) + \varphi(d(Tp, p))] \\
 &= [\beta_n + (1 - \beta_n)a] d(x_{n-1}, p) \tag{3.2}
 \end{aligned}$$

Using (4.2) in (4.1), we have

$$d(x_n, p) \leq [\alpha_n + (1 - \alpha_n)a] [\beta_n + (1 - \beta_n)a] d(x_{n-1}, p) \tag{3.3}$$

Let  $\delta = [\alpha_n + (1 - \alpha_n)a] [\beta_n + (1 - \beta_n)a]$

$$= \left( \sum_{j=0}^1 \alpha_{n,j} a^j \right) \left( \sum_{j=0}^1 \beta_{n,j} a^j \right)$$

where  $\alpha_n = \alpha_{n,0}, (1 - \alpha_n) = \alpha_{n,1}, \beta_n = \beta_{n,0}, (1 - \beta_n) = \beta_{n,1}$  and  $a^j \in [0, 1)$  for  $j = 0, 1$ . We have,

$$\delta = \left( \sum_{j=0}^1 \alpha_{n,j} a^j \right) \left( \sum_{j=0}^1 \beta_{n,j} a^j \right) < \left( \sum_{j=0}^1 \alpha_{n,j} \right) \left( \sum_{j=0}^1 \beta_{n,j} \right) = 1$$

Hence, by Lemma 1, we have  $\lim_{n \rightarrow \infty} d(x_n, p) = 0$ .

Therefore, the Thianwan-iteration (2.1) converges strongly to  $p \in F(T)$ .

**THEOREM 3.2.2** Let  $K$  be a nonempty closed subset of a convex metric space  $X$  and  $T$  be a self map of  $K$  satisfying the contractive-type condition (2.10) with  $F(T) \neq \emptyset$ . Let  $x_n \in K$  and  $\{x_n\}$  be a sequence defined by (2.2), then

- (i) the mapping  $T$  satisfying (2.10) has a unique fixed point.
- (ii) the sequence defined by (2.2) converges strongly to the fixed point  $p \in F(T)$ .

**PROOF**

(i) Suppose  $p_1, p_2 \in F(T)$  and  $p_1 - p_2 \neq 0$ , then by (2.10)

$$d(p_1, p_2) = d(Tp_1, Tp_2) \leq ad(p_1, p_2) + \varphi(d(Tp_1, p_2))$$

Implying that  $d(p_1, p_2) \leq ad(p_1, p_2)$  and  $(1 - a)d(p_1, p_2) \leq 0$

Since  $(1 - a)$  is positive, then  $d(p_1, p_2) < 0$  is false.

Hence,  $d(p_1, p_2) = 0$  if and only if  $p_1 = p_2$ .

Therefore,  $T$  has a unique fixed point.

(ii) Suppose  $x_0 \in K$  and  $p \in F(T)$ , by (2.2) and (2.10) we have

$$\begin{aligned}
 d(x_n, p) &= d(W(x_{n-1}, Ty_{n-1}, \alpha_n), p) \leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) d(Ty_{n-1}, p) \\
 &\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) ad(y_{n-1}, p) + (1 - \alpha_n) \varphi(d(Tp, p)) = \\
 &\alpha_n d(x_{n-1}, p) + (1 - \alpha_n) ad(y_{n-1}, p) \tag{3.4}
 \end{aligned}$$

Also, from (2.2), we have

$$\begin{aligned}
 d(y_{n-1}, p) &= d \leq \beta_n d(x_{n-1}, p) + (1 - \beta_n) d(Tx_{n-1}, p) \\
 &\leq \beta_n d(x_{n-1}, p) + (1 - \beta_n) ad(x_{n-1}, p) = [\beta_n + (1 - \beta_n)a] d(y_{n-1}, p) \tag{3.5}
 \end{aligned}$$

Substituting (3.5) into (3.4) becomes

$$d(x_n, p) \leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n)a [\beta_n + (1 - \beta_n)a]d(x_{n-1}, p) =$$

$$[\alpha_n + (1 - \alpha_n)a\beta_n + (1 - \alpha_n)(1 - \beta_n)a^2]d(x_{n-1}, p)$$

(3.6)

Let,

$$\delta = \alpha_n + (1 - \alpha_n)a\beta_n + (1 - \alpha_n)(1 - \beta_n)a^2$$

$$< \alpha_n + (1 - \alpha_n)\beta_n + (1 - \alpha_n)(1 - \beta_n) = 1$$

By the application of Lemma 1 to (3.6), we have

$$\lim_{n \rightarrow \infty} x_n = p$$

Therefore,  $\{x_n\}$  converges strongly to  $p$ .

**COROLLARY 1** Let  $(X, d, W)$  be a convex metric space and  $K$  be a closed subset of  $X$ . Let  $T$  be a self-map of  $K$  satisfying (2.10) with  $F(T) \neq \emptyset$ . For  $x_0 \in K$ , then,

- (i) the mapping  $T$  satisfying (2.10) has a unique fixed point.
- (ii) the sequence  $\{x_n\}$  defined by (2.5) converges strongly to the fixed point  $p \in F(T)$ .

The proof of this Corollary is direct from Theorem 3.2.1 and Theorem 3.2.2 when  $\beta_n = 1$  in (2.1) and (2.2). The stability results are as follows:

**THEOREM 3.2.3** Let  $(X, d, W)$  be a convex metric space and  $K$  a nonempty closed subset of  $(X, d, W)$ . Suppose  $T$  is a map of  $K$  into itself and satisfies the condition (2.10) with  $F(T) \neq \emptyset$ . Then, for  $x_0 \in K$ , the sequence  $\{x_n\}$  defined by (2.1) is  $T$ -stable.

**PROOF**

Let  $\{p_n\}$  be an arbitrary sequence in  $K$  and set  $\epsilon_n = d(p_n, W(q_{n-1}, Tq_{n-1}, \alpha_n))$ , where  $q_{n-1} = W(p_{n-1}, Tp_{n-1}, \beta_n)$ . Suppose  $\lim \epsilon_n = 0$  and  $p \in F(T)$ , then

$$d(p_n, p) \leq d(p_n, W(q_{n-1}, Tq_{n-1}, \alpha_n)) + d(W(q_{n-1}, Tq_{n-1}, \alpha_n), p)$$

$$\leq \epsilon_n + \alpha_n d(q_{n-1}, p) + (1 - \alpha_n)d(Tq_{n-1}, p)$$

$$\leq \epsilon_n + \alpha_n d(q_{n-1}, p) + (1 - \alpha_n)a d(q_{n-1}, p)$$

$$\leq \epsilon_n + [\alpha_n + (1 - \alpha_n)a] [\beta_n + (1 - \beta_n)a] d(q_{n-1}, p)$$

Since  $\lim \epsilon_n = 0$  and  $[\alpha_n + (1 - \alpha_n)a] [\beta_n + (1 - \beta_n)a] < 1$ , then

$$\lim_{n \rightarrow \infty} p_n = p$$

Conversely, suppose  $\lim_{n \rightarrow \infty} p_n = p$ , then

$$\epsilon_n = d(p_n, W(q_{n-1}, Tq_{n-1}, \alpha_n))$$

$$\leq d(p_n, p) + d(p, W(q_{n-1}, Tq_{n-1}, \alpha_n))$$

$$\leq d(p_n, p) + \alpha_n d(q_{n-1}, p) + (1 - \alpha_n)a d(q_{n-1}, p)$$

$$\leq d(p_n, p) + [\alpha_n + (1 - \alpha_n)a] [\beta_n + (1 - \beta_n)a] d(p_{n-1}, p)$$

Since  $d(p_n, p) \rightarrow 0$ , then  $d(p_{n-1}, p) \rightarrow 0$  and hence,

$$\lim_{n \rightarrow \infty} \epsilon_n = 0$$

Therefore, the scheme (2.1) is T-stable.

**THEOREM 3.2.4** Let  $K$  be a nonempty closed subset of a convex metric space  $(X, d, W)$  and  $T$  be contrative-type operator satisfying (2.10) with  $F(T) \neq \emptyset$ .

Then, for  $x_0 \in K$ , the sequence  $\{x_n\}$  defined by (2.2) is  $T$ -stable.

**PROOF**

Let  $\epsilon_n = d(p_n, W(p_{n-1}, Tq_{n-1}, \alpha_n))$  for  $\{p_n\} \subset K$  an arbitrary sequence, where  $q_{n-1} = W(p_{n-1}, Tp_{n-1}, \beta_n)$ .

Suppose  $\lim \epsilon_n = 0$  and  $p \in F(T)$ , then

$$\begin{aligned} d(p_n, p) &\leq d(p_n, W(p_{n-1}, Tq_{n-1}, \alpha_n)) + d(W(p_{n-1}, Tq_{n-1}, \alpha_n), p) \\ &\leq \epsilon_n + \alpha_n d(p_{n-1}, p) + (1 - \alpha_n) d(Tq_{n-1}, p) \\ &\leq \epsilon_n + \alpha_n d(p_{n-1}, p) + (1 - \alpha_n) \alpha d(q_{n-1}, p) \end{aligned}$$

$$\leq \epsilon_n + \alpha_n + (1 - \alpha_n) \alpha \beta_n + (1 - \alpha_n) (1 - \beta_n) \alpha^2 d(p_{n-1}, p)$$

Since  $\lim \epsilon_n = 0$  and  $\alpha_n + (1 - \alpha_n) \alpha \beta_n + (1 - \alpha_n) (1 - \beta_n) \alpha^2 < 1$ , then

$$\lim_{n \rightarrow \infty} p_n = p$$

Conversely, suppose  $\lim_{n \rightarrow \infty} p_n = p$ , then

$$\begin{aligned} \epsilon_n &= d(p_n, W(p_{n-1}, Tq_{n-1}, \alpha_n)) \\ &\leq d(p_n, p) + d(W(p_{n-1}, Tq_{n-1}, \alpha_n), p) \\ &\leq d(p_n, p) + \alpha_n d(p_{n-1}, p) + (1 - \alpha_n) \alpha d(q_{n-1}, p) \\ &\leq d(p_n, p) + \alpha_n + (1 - \alpha_n) \alpha \beta_n + (1 - \alpha_n) (1 - \beta_n) \alpha^2 d(p_{n-1}, p) \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} p_n = p$ , then,

$$\lim_{n \rightarrow \infty} \epsilon_n = 0$$

Therefore, the scheme (3.2) is T-stable.

**THEOREM 3.2.5** Let  $(X, d, W)$  be a convex metric space and  $K$  a nonempty closed subset of  $(X, d, W)$ . Suppose  $T$  is a map of  $K$  into itself and satisfies the condition (2.10) with  $F(T) \neq \emptyset$ . Then, for  $x_0 \in K$ , the sequence  $\{x_n\}$  defined by (2.5) is  $T$ -stable.



**PROOF**

Let  $\{p_n\}$  be an arbitrary sequence in  $K$  and set  $\epsilon_n = d(p_n, W(p_{n-1}, Tp_{n-1}, \alpha_n))$

Suppose  $\lim \epsilon_n = 0$  and  $p \in F(T)$ , then

$$\begin{aligned} d(p_n, p) &\leq d(p_n, W(p_{n-1}, Tp_{n-1}, \alpha_n)) + d(W(p_{n-1}, Tp_{n-1}, \alpha_n), p) \\ &\leq \epsilon_n + \alpha_n d(p_{n-1}, p) + (1 - \alpha_n) d(Tp_{n-1}, p) \\ &\leq \epsilon_n + \alpha_n d(p_{n-1}, p) + (1 - \alpha_n) a d(p_{n-1}, p) \\ &\leq \epsilon_n + [\alpha_n + (1 - \alpha_n)a] d(p_{n-1}, p) \end{aligned}$$

Since  $\lim \epsilon_n = 0$  and  $[\alpha_n + (1 - \alpha_n)a] < 1$ , then

$$\lim_{n \rightarrow \infty} p_n = p$$

Conversely, suppose  $\lim_{n \rightarrow \infty} p_n = p = p$ , then

$$\begin{aligned} \epsilon_n &= d(p_n, W(p_{n-1}, Tp_{n-1}, \alpha_n)) \\ &\leq d(p_n, p) + d(p, W(p_{n-1}, p_{n-1}, \alpha_n)) \\ &\leq d(p_n, p) + \alpha_n d(p_{n-1}, p) + (1 - \alpha_n) a d(p_{n-1}, p) \\ &\leq d(p_n, p) + [\alpha_n + (1 - \alpha_n)a] d(p_{n-1}, p) \end{aligned}$$

Since  $d(p_n, p) \rightarrow 0$ , then  $d(p_{n-1}, p) \rightarrow 0$ , and hence,

$$\lim_{n \rightarrow \infty} \epsilon_n = 0$$

Therefore, the scheme (2.5) is T-stable.

**Numerical Examples**

We compare the convergence rate of the three schemes with the following two examples:

**Example 1** Let  $f : [\frac{1}{2}, 2] \rightarrow [\frac{1}{2}, 2]$  be defined by  $f(x) = \frac{1}{x}$ , an oscillatory function with  $p = 1$ . initial point  $x_0 = 4$  and  $\alpha_n = \beta_n = \frac{1}{n+1}$ .

**Example 2** Let  $f : [6, 8] \rightarrow [6, 8]$  be defined by  $f(x) = \frac{x}{2} + 3$ , an increasing function with fixed point  $p = 6$  and initial guess  $x_0 = 7$  using  $\alpha_n = \beta_n = \frac{1}{\sqrt{5n+1}}$ .

**Solutions:** The solutions to the two examples are computed using MATLAB and are presented in tables 1 and 2.

**TABLE 1:** Results for Example 1

N	Mann iteration	Ishikawa iteration	Thianwan iteration
1	2.1250	3.0625	1.2978
2	1.0221	1.8465	1.0199
3	0.9893	1.1125	1.0049
4	1.0065	0.9758	1.0018
5	0.9957	1.0098	1.0008
6	1.0031	0.9955	1.0004
7	0.9977	1.0024	1.0002

8	1.0018	0.9986	1.0001
9	0.9986	1.0009	1.0001
10	1.0012	0.9994	1.0001
11	0.9990	1.0004	1.0001
12	1.0008	0.9997	1.0001
13	0.9993	1.0002	1.0001
14	1.0006	0.9999	1.0001
15	0.9995	1.0001	1.0001
...	...	...	...
28	1.0002	0.9999	1.0001
29	0.9999	1.0001	1.0001
30	1.0001	0.9999	1.0001

**Remark 1:** It is observed, from Table 1 of the function  $f(x) = \frac{1}{x}$  with  $\alpha_n = \beta_n = \frac{1}{n+1}$ , that the Thianwan, Ishikawa and Mann iterations converge in 8, 14 and 29 iterations respectively. Therefore, Thianwan-iteration is faster in terms of convergence rate than Ishikawa and Mann iterations.

**Table 2:** Results for Example 2

N	Mann iteration	Ishikawa iteration	Thianwan iteration
1	6.7500	6.6166	6.5625
2	6.5167	6.3260	6.2670
3	6.1405	6.1579	6.1156
4	6.0897	6.0721	6.0472
5	6.0561	6.0315	6.0184
6	6.0345	6.0133	6.0070
7	6.0209	6.0054	6.0026
8	6.0125	6.0022	6.0009
9	6.0075	6.0009	6.0003
10	6.0044	6.0003	6.0001
11	6.0026	6.0001	6.0000
12	6.0015	6.0000	6.0000
...	...	...	...
18	6.0001	6.0000	6.0000
19	6.0000	6.0000	6.0000
20	6.0000	6.0000	6.0000

**Remark 2** Here in Table 2 of the function  $f(x) = \frac{x}{2} + 3$ , an increasing function with fixed point  $p = 6$  and initial guess  $x_0 = 7$  using  $\alpha_n = \beta_n = \frac{1}{\sqrt{5n+1}}$ . It is observed that the Thianwan, Ishikawa and Mann iterations converge at 11, 12 and 19 iterations respectively.

## Conclusion

The prove of strong convergence and stability results of Thianwan-iteration, Ishikawa iteration and Mann iteration for quasi-contractive operator in convex metric spaces were considered. Our results show that the Thianwan-iteration converges faster than both Ishikawa and Mann iterations. Furthermore, Ishikawa iteration is faster than Mann iteration in terms of convergence rate.

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