Comparative Analysis of Some Stability Regions of a Class of Block Integrators for the Solution of First-Order Ordinary Differential Equations

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Abstract

In this paper, we compare the stability regions of two block integrators developed by Odekunle, Adesanya and Sunday (2012A, 2012B) for the solution of first-order ordinary differential equations. The approximate solution used in deriving these integrators is a combination of power series and exponential function. This analysis is made basically to buttress the fact that one integrator tends to perform better than the other if the latter has a larger stability region than the former. Some basic properties of the two block integrators are further examined.

Keywords: *Approximate Solution, Block Integrator, Exponential Function, Power Series,* Stability *Region*
AMS Subject Classification: 65L05, 65L06, 65D30

Introduction

One block integrator is more stable than the other if the latter has a larger Region of Absolute Stability (RAS) than the former. In this paper, we compare the stability region of the block integrator developed by Odekunle, Adesanya and Sunday (2012A) with that of Odekunle, Adesanya and Sunday (2012B). Both integrators were derived for the solution of,

$$y' = f(x,y),\ y(a) = \eta\ \forall\ a \le x \le b \quad \text{……..(1)}$$

where f is continuous within the interval of integration $[a,b]$. We assume that f satisfies Lipchitz condition which guarantees the existence and uniqueness of solution of (1). The problem (1) occurs mainly in the study of dynamical systems and electrical networks. According to Kandasamy *et al.* (2005) and Sunday (2011), equation (1) is used in simulating the growth of populations, trajectory of a particle, simple harmonic motion, deflection of a beam etc.

Development of Linear Multistep Methods (LMMs) for solving ODEs can be generated using methods such as Taylor's series, numerical integration and collocation method, which are restricted by an assumed order of convergence, Ehigie *et al.* (2011).

Block integrators for solving ODEs have initially been proposed by Milne (1953) who used them as starting values for predictor-corrector algorithm, Rosser (1967) developed Milne's method in form of implicit integrators, and Shampine and Watts (1969) also contributed greatly to the development and application of block integrators. More recently, authors like Butcher (2003), Zarina *et al.* (2005), Awoyemi *et al.* (2007), Areo *et al.* (2011), Ibijola *et al*. (2011), Chollom *et al.* (2012), have all proposed LMMs to generate numerical solution to (1). These authors proposed integrators in which the approximate solution ranges from power series, Chebychev's, Lagrange's and Laguerre's polynomials. The advantages of LMMs over single step methods have been extensively discussed by Awoyemi (2001).

The Two Block Integrators

In deriving the two block integrators, interpolation and collocation procedures were used by choosing interpolation point s at a grid point and collocation points r at all points giving rise to $\xi = s + r - 1$ system of equations whose coefficients are determined by using appropriate procedures. The approximate solution to (1) is taken to be a combination of power series and exponential function. For Odekunle, Adesanya and Sunday (2012A), the approximate solution is,

$$y(x) = \sum_{j=0}^{3} a_j x^j + a_4 \sum_{j=0}^{4} \frac{\alpha^j x^j}{j!} \quad \ldots\ldots\ldots\ldots..(2)$$

while for Odekunle, Adesanya and Sunday (2012B), the approximate solution is,

$$y(x) = \sum_{j=0}^{4} a_j x^j + a_5 \sum_{j=0}^{5} \frac{\alpha^j x^j}{j!} \quad \ldots\ldots\ldots\ldots...(3)$$

where $x \in [a, b]$, the $a's$ are real unknown parameters to be determined and α is a real number.

They arrived at the block integrators of the form,

$$A^{(0)}\mathbf{Y}_m = \mathbf{E}\mathbf{y}_n + hd\mathbf{f}(\mathbf{y}_n) + hb\mathbf{F}(\mathbf{Y}_m) \quad \ldots\ldots.(4)$$

where for Odekunle, Adesanya and Sunday (2012A),

$$\mathbf{Y}_m = \left[y_{n+1}, y_{n+2}, y_{n+3}\right]^T, \mathbf{y}_n = \left[y_{n-2}, y_{n-1}, y_n\right]^T$$

$$\mathbf{F}(\mathbf{Y}_m) = \left[f_{n+1}, f_{n+2}, f_{n+3}\right]^T, \mathbf{f}(\mathbf{y}_n) = \left[f_{n-2}, f_{n-1}, f_n\right]^T$$

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{E} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ d = \begin{bmatrix} 0 & 0 & \frac{9}{24} \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{9}{24} \end{bmatrix}$$

$$b = \begin{bmatrix} \frac{19}{24} & \frac{-5}{24} & \frac{1}{24} \\ \frac{4}{3} & \frac{1}{3} & 0 \\ \frac{27}{24} & \frac{27}{24} & \frac{9}{24} \end{bmatrix}$$

and for Odekunle, Adesanya and Sunday (2012B),

$$\mathbf{Y}_m = [y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}]^T, \mathbf{y}_n = [y_{n-3}, y_{n-2}, y_{n-1}, y_n]^T,$$
$$\mathbf{F}(\mathbf{Y}_m) = [f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}]^T, \mathbf{f}(\mathbf{y}_n) = [f_{n-3}, f_{n-2}, f_{n-1}, f_n]^T,$$

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{E} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ d = \begin{bmatrix} 0 & 0 & 0 & \frac{251}{720} \\ 0 & 0 & 0 & \frac{29}{90} \\ 0 & 0 & 0 & \frac{27}{80} \\ 0 & 0 & 0 & \frac{14}{45} \end{bmatrix}$$

$$b = \begin{bmatrix} \frac{323}{360} & \frac{-11}{30} & \frac{53}{360} & \frac{-19}{720} \\ \frac{62}{45} & \frac{4}{15} & \frac{2}{45} & \frac{-1}{90} \\ \frac{51}{40} & \frac{9}{10} & \frac{21}{40} & \frac{-3}{80} \\ \frac{64}{45} & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} \end{bmatrix}$$

Zero-Stability of the Two Block Integrators

Definition 1

The block integrator (4) is said to be zero-stable, if the roots $z_s, s = 1, 2, ..., k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z) = \det(z\mathbf{A}^{(0)} - \mathbf{E})$ satisfies $|z_s| \le 1$ and every root satisfying $|z_s| \le 1$ have multiplicity not exceeding the order of the differential equation. Moreover, as $h \to 0$, $\rho(z) = z^{r-\mu}(z-1)^{\mu}$ where μ is the order of the differential equation, r is the order of the matrices $\mathbf{A}^{(0)}$ *and* $\mathbf{E}$ (see Awoyemi *et al.* (2007) for details).

For Odekunle, Adesanya and Sunday (2012A),

$$\rho(z) = \left| z \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

$$=\begin{vmatrix} z & 0 & -1 \\ 0 & z & -1 \\ 0 & 0 & z-1 \end{vmatrix} = z^2(z-1) \qquad (5)$$

Solving for z in (5) gives $z=0\, or\, z=1$. Hence, the block integrator is zero-stable. Similarly, for Odekunle, Adesanya and Sunday (2012B),

$$\rho(z) = \left| z\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$=\begin{vmatrix} z & 0 & 0 & -1 \\ 0 & z & 0 & -1 \\ 0 & 0 & z & -1 \\ 0 & 0 & 0 & z-1 \end{vmatrix} = z^3(z-1) \qquad (6)$$

Thus, solving for z in (6) gives $z=0\, or\, z=1$. Hence, the block integrator is also zero-stable.

STABILITY REGIONS OF THE TWO BLOCK INTEGRATORS

To determine the absolute stability regions of the two block integrators, we adopt the boundary locus method. This is achieved by substituting the test equation,

$y' = -\lambda y$ ……….(7)

into the block formula (4). This gives,

$$\mathbf{A}^{(0)}\mathbf{Y}_m(r) = \mathbf{E}\mathrm{y}_n(r) - h\lambda\mathbf{D}y_n(r) - h\lambda\mathbf{B}\mathbf{Y}_m(r) \quad …..(8)$$

Thus,

$$\bar{h}(r) = -\left(\frac{\mathbf{A}^{(0)}Y_m(r) - \mathbf{E}y_n(r)}{\mathbf{D}y_n(r) + \mathbf{B}Y_m(r)}\right) \quad ………(9)$$

Writing (9) in trigonometric ratios gives,

$$\bar{h}(\theta) = -\left(\frac{\mathbf{A}^{(0)}Y_m(\theta) - \mathbf{E}y_n(\theta)}{\mathbf{D}y_n(\theta) + \mathbf{B}Y_m(\theta)}\right) \quad ……..(10)$$

where $r = e^{i\theta}$. Equation (10) is our characteristic or stability polynomial.

Applying (10) to the block integrators developed by Odekunle, Adesanya and Sunday (2012A) and (2012B), we obtain the stability polynomials,

$$\bar{h}_{(2012A)}(\theta) = \frac{(\cos 2\theta)(\cos\theta) - (\cos 2\theta)(\cos 3\theta)(\cos\theta)}{\frac{1}{4}(\cos 2\theta)(\cos\theta) - \frac{1}{4}(\cos 2\theta)(\cos 3\theta)(\cos\theta)} \quad ……………………….(11)$$

and

$$\bar{h}_{(2012B)}(\theta)=\frac{(\cos 2\theta)(\cos 3\theta)(\cos\theta)-(\cos 2\theta)(\cos 3\theta)(\cos 4\theta)(\cos\theta)}{\frac{1}{5}(\cos 2\theta)(\cos 3\theta)(\cos\theta)-\frac{1}{5}(\cos 2\theta)(\cos 3\theta)(\cos 4\theta)(\cos\theta)} \qquad \ldots\ldots\ldots..(12)$$

respectively. The table below summarizes other basic properties of the two block integrators.

Table 1: Comparism of Some Basic Properties of the Two Block Integrators

S/No.	Block Integrators	Order	Consistence	Convergence
1.	Odekunle, Adesanya and Sunday (2012A)	4	Consistent	Convergent
2.	Odekunle, Adesanya and Sunday (2012B)	5	Consistent	Convergent

The stability regions of the two block integrators are as shown in the figures below.

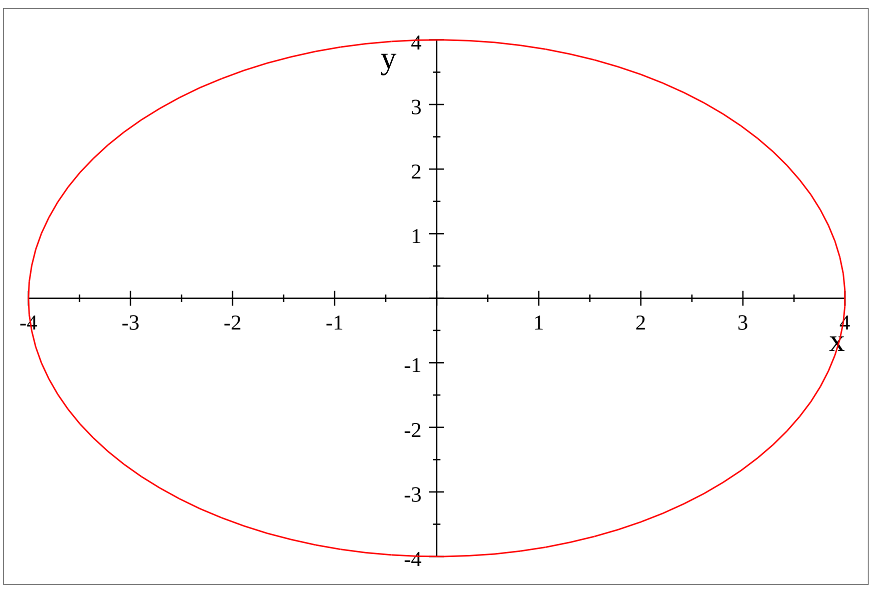

Fig. 1: Stability Region of Odekunle, Adesanya and Sunday (2012A)

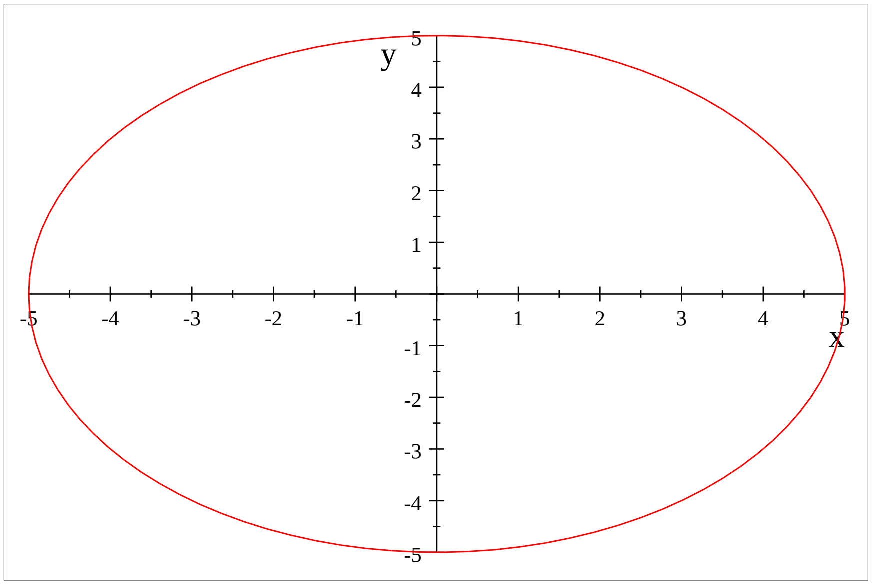

Fig. 2: Stability Region of Odekunle, Adesanya and Sunday (2012B)

From the 3D point of view, the RAS of the two block integrators are given by;

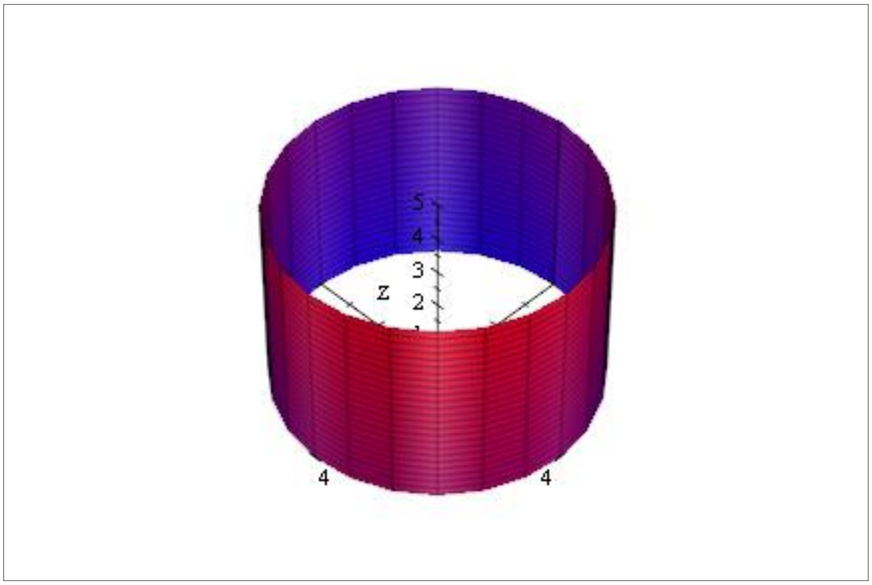

Fig. 3: Stability Region of Odekunle, Adesanya and Sunday (2012A)

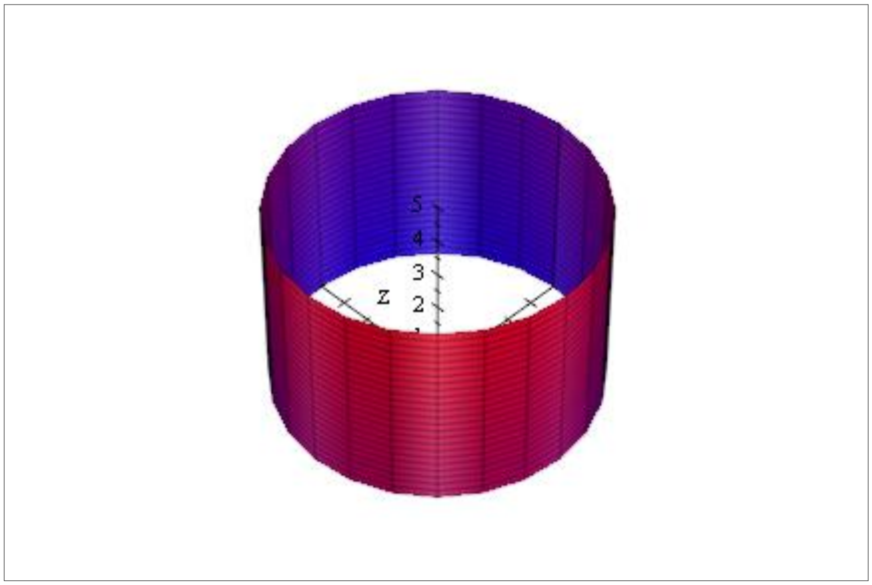

Fig. 4: Stability Region of Odekunle, Adesanya and Sunday (2012B)

Conclusion

Finally, it is clear from the presentation above that the stability region of Odekunle, Adesanya and Sunday (2012A) is a sub-region of that of Odekunle, Adesanya and Sunday (2012B). Little wonder, the performance of the latter is better than that of the former on problems of the form (1). See Odekunle, Adesanya and Sunday (2012A, 2012B) for details on numerical implementations.

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