©Adamawa State University Journal of Scientific Research. ISSN: 2251-0702

TETFUND/UNIMUBI/ARJ/3

A FAMILY OF BLOCK GENERALIZED MILNE-SIMPSON METHODS FOR THE SOLUTION OF INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

Kumleng G. M.[1], Skwame Y. [2] and Kutchin S.Y[1]

[1]Department of Mathematics, University of Jos, Plateau State, Nigeria

[2]Department of Mathematics, Adamawa State University, Mubi, Nigeria.

ABSTRACT

This paper is focused on the construction and accurate implementation of the block generalized Milne-Simpson methods for step numbers k = 2, 3 and 4. The methods compute the solution of systems of initial value problems in ordinary differential equations on non- overlapping subintervals. The block methods derived are all convergent and the plot of their absolute stability regions shows that they are A-stable. The accuracy of the block methods is illustrated using some stiff systems of ODEs.

KEYWORDS: A-Stability, Block Multistep Method, Initial value Problems, Ordinary Differential Equations

INTRODUCTION

Consider the numerical solution of the first order initial value problem of the form,

$$y' = f(x, y), \quad y(x_0) = y_0, \quad x \in [a, b], \;\; y \in R \tag{1}$$

where f is continuous and satisfies Lipchitz's condition which guarantees that a unique solution exist. The development of numerical methods for approximating the solution of (1) has in recent years received a great deal of attention and as such many researchers have constructed efficient methods for its solution.(Onumanyi, *et al.* 1994) in a bid to solve (1) constructed block methods using the multistep collocation method. These block methods share similar properties with the Runge-Kutta method for being self-starting and do not require special predictors or starting values. Milne (1953) introduced the idea of block methods to generate starting values for predictor-corrector algorithms for solving ODEs. Authors who have also contributed in the theory of block methods include. (Ajie *et al.* 2014; Bond *et al.* 1979; Fatunla,1991; Kumleng, *et al.*2013; Henrici,1962; Nasir, *et al.* 2011; Onumanyi, *et al.* 1994; Rosser,1967; Shampine, *et al.* 1972).

In this paper, the construction of a new family of block generalized Milne-Simpson methods for step numbers k = 2, 3, and 4 is considered. The continuous formulations of these block methods are obtained using the multistep collocation method of (Onumanyi, *et al.* 1994). These formulations evaluated at some desired grid points gives discrete schemes which are combined as block methods for the simultaneous integration of IVP's without requiring starting values.

FORMULATION OF THE METHOD

The solution of the initial value problem in (1) is assumed to be the polynomial

$$y(x) = \sum_{j=0}^{s+r+1} a_j x^j \qquad (2)$$

Where s and r are the number of interpolation and collocation points respectively. From (2) we have

$$y'(x) = \sum_{j=0}^{s+r-1} j a_j x^{j-1} \qquad (3)$$

Substituting (3) into (1) gives

$$f(x,y) = \sum_{j=0}^{s+r-1} j a_j x^{j-1} \qquad (4)$$

Collocating (3) at x_{n+s}, $s = 0,1,2$ and interpolating (2) at $x = x_n$, gives a linear system of equations in the form:

$$AX = U \qquad (5)$$

where

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 \\ 0 & 1 & 2x_n & 3x_n^2 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 \end{bmatrix}, \quad A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T, U = \begin{bmatrix} y_n \\ f_n \\ f_{n+1} \\ f_{n+2} \end{bmatrix}$$

Solving (5) for a_0, a_1, a_2, a_3 and substituting the result into (2) gives the continuous form of the two step method as:

$$y(x) = a_0 y_n + h \sum_{j=0}^{2} \beta_j(x) f_{n+j} \qquad (6)$$

where $a_0 = 1$

$$h\beta_0(x) = \left(\tau - \frac{3}{4}\frac{\tau^2}{h} + \frac{1}{6}\frac{\tau^3}{h^2} \right), \ h\beta_1(x) = \left(\frac{\tau^2}{h} - \frac{1}{3}\frac{\tau^3}{h^2} \right)$$

$$h\beta_2(x) = \left(-\frac{1}{4}\frac{\tau^2}{h} + \frac{1}{6}\frac{\tau^3}{h^2} \right)$$

where $\tau = x - x_n$, $\tau \in [0, 2h]$.

Now evaluating (6) at the points $\tau = h, 2h$ gives the two discrete methods which constitute the two step block generalized Milne Simpson method as:

$$\begin{aligned} y_{n+1} - y_n &= \frac{h}{12}\left(5f_n + 8f_{n+1} - f_{n+2}\right) \\ y_{n+2} - y_n &= \frac{h}{3}\left(f_n + 4f_{n+1} + f_{n+2}\right) \end{aligned} \tag{7}$$

Similarly, we obtain the continuous form of the three step generalized Milne Simpson method as:

$$y(x) = a_1 y_{n+1} + h\sum_{j=0}^{3}\beta_j(x) f_{n+j} \tag{8}$$

where $a_1 = 1$

$$\beta_0(x) = \left(-\frac{3}{8}h + \tau - \frac{11}{12}\frac{\tau^2}{h} + \frac{1}{3}\frac{\tau^3}{h^2} - \frac{1}{24}\frac{\tau^4}{h^3} \right)$$
$$\beta_1(x) = \left(\frac{3}{2}\frac{\tau^2}{h} - \frac{19}{24}h - \frac{5}{6}\frac{\tau^3}{h^2} + \frac{1}{8}\frac{\tau^4}{h^3} \right)$$
$$\beta_2(x) = \left(\frac{5}{24}h - \frac{3}{4}\frac{\tau^2}{h} + \frac{2}{3}\frac{\tau^3}{h^2} - \frac{1}{8}\frac{\tau^4}{h^3} \right)$$
$$\beta_3(x) = \left(\frac{1}{6}\frac{\tau^2}{h} - \frac{1}{6}\frac{\tau^3}{h^2} + \frac{1}{24}\frac{\tau^4}{h^3} - \frac{1}{24}h \right)$$

where $\tau = x - x_n$, $\tau \in [0, 3h]$.

Now evaluating (8) at the points $\tau = 0, 2h, 3h$ gives the three discrete methods which constitute the three step block generalized Milne Simpson method as:

$$y_{n+1} - y_n = \frac{h}{24}\left(9f_n + 19f_{n+1} - 5f_{n+2} + f_{n+3}\right)$$

$$y_{n+2} - y_{n+1} = \frac{1}{24} h\left(-f_n + 13f_{n+1} + 13f_{n+2} - f_{n+3}\right) \tag{9}$$
$$y_{n+3} - y_{n+1} = \frac{h}{3}\left(f_{n+1} + 4f_{n+2} + f_{n+3}\right)$$

The continuous form of the four step generalized Milne Simpson method is obtained as:

$$y(x) = a_2 y_{n+2} + h\sum_{j=0}^{4} \beta_j(x) f_{n+j} \tag{10}$$

where $a_2 = 1$

$$\beta_0(x) = \left(-\frac{29}{90}h + \tau - \frac{25}{24}\frac{\tau^2}{h} + \frac{35}{72}\frac{\tau^3}{h^2} - \frac{5}{48}\frac{\tau^4}{h^3} + \frac{1}{120}\frac{\tau^5}{h^4}\right)$$
$$\beta_1(x) = \left(\frac{2\tau^2}{h} - \frac{13}{9}\frac{\tau^3}{h^2} + \frac{3}{8}\frac{\tau^4}{h^3} - \frac{62}{45}h - \frac{1}{30}\frac{\tau^5}{h^4}\right)$$
$$\beta_2(x) = \left(-\frac{3}{2}\frac{\tau^2}{h} + \frac{19}{12}\frac{\tau^3}{h^2} + \frac{1}{20}\frac{\tau^5}{h^4} - \frac{4}{15}h - \frac{1}{2}\frac{\tau^4}{h^3}\right)$$
$$\beta_3(x) = \left(\frac{2}{3}\frac{\tau^2}{h} - \frac{2}{45}h - \frac{1}{30}\frac{\tau^5}{h^4} - \frac{7}{9}\frac{\tau^3}{h^2} + \frac{7}{24}\frac{\tau^4}{h^3}\right)$$
$$\beta_4(x) = \left(\frac{1}{90}h - \frac{1}{8}\frac{\tau^2}{h} + \frac{11}{72}\frac{\tau^3}{h^2} + \frac{1}{120}\frac{\tau^5}{h^4} - \frac{1}{16}\frac{\tau^4}{h^3}\right)$$

where $\tau = x - x_n$, $\tau \in [0, 4h]$.

Now evaluating (10) at the points $\tau = 0,\ h, 3h,\ 4h$ gives the four discrete methods which constitute the four-step block generalized Milne Simpson method as:

$$y_n - y_{n+2} = \frac{h}{90}\left(-29f_n - 124f_{n+1} - 24f_{n+2} - 4f_{n+3} + f_{n+4}\right)$$
$$y_{n+1} - y_{n+2} = \frac{h}{720}\left(19f_n - 346f_{n+1} - 456f_{n+2} + 74f_{n+3} - 11f_{n+4}\right) \tag{11}$$

$$y_{n+3} - y_{n+2} = \frac{h}{720}\left(11f_n - 74f_{n+1} + 456f_{n+2} + 346f_{n+3} - 19f_{n+4}\right)$$
$$y_{n+4} - y_{n+2} = \frac{1}{90} h\left(-f_n + 4f_{n+1} + 24f_{n+2} + 124f_{n+3} + 29f_{n+4}\right)$$

Convergence analysis of our new Block Methods

Order and Error constants of the block methods

Following Lambert (1973) and Fatunla (1991), the local truncation error associated with the general linear multistep method is the linear difference operator L defined as:

$$L[y(x),h]=\sum_{j=0}^{k}\{\alpha_j y(x_{n+j})-\beta_j y'(x_{n+j}\} \tag{12}$$

Assuming that $y(x)$ is sufficiently differentiable, we expand (12) as a Taylor series about the point x_n to obtain the expression:

$$L[y(x),h]=C_0y(x_n)+C_1hy'(x_n)+...+C_qh^qy^q(x_n)+...\cong 0 \tag{13}$$

where the constant C_q, $q=0,1,...$ are given as

$$C_0=\sum_{j=0}^{k}\alpha_j\,,\ \ C_1=\sum_{j=0}^{k}j\alpha_j \tag{14}$$

$$C_q=\tfrac{1}{q!}(\alpha_1+2^q\alpha_2+...+k^q\alpha_k)-\tfrac{1}{(q-1)!}(\beta_1+2^{q-1}\beta_2+...+k^{q-1}\beta_k) \tag{15}$$

According to (13), the general linear multistep method has order p if $C_0=C_1=...=C_{p-1}=C_p=0, \text{ and } C_{p+1}\neq 0.$ C_{p+1} is the error constant of the method and $C_{p+1}h^{p+1}y^{p+1}(x_n)$ is the principal local truncation error at the point x_n. Thus the local truncation error (LTE) is written as:

$$LTE=C_{p+1}h^{p+1}y^{p+1}(x_n)+O(h^{p+2}) \tag{16}$$

Table 1: Order and error constants of the block methods (6),(9),(11)

Method	Order P	Error constant C_{p+1}
(6)	$[3,3]^T$	$[\frac{1}{24},-\frac{1}{90}]^T$
(9)	$[4,4,4]^T$	$[-\frac{19}{720},\frac{11}{720},-\frac{1}{90}]^T$
(11)	$[5,5,5,5]^T$	$[-\frac{1}{90},\frac{11}{1440},\frac{11}{1440},-\frac{1}{90}]^T$

Since the order of each of the block methods is greater than one, we conclude that the block method is consistent.

Zero Stability

Definition: A block linear multistep method is said to be zero-stable if the roots of its first characteristic polynomial $\rho(\lambda)$ defined by $\rho(\lambda)=\det(\lambda A^{(0)}-A^{(1)})$ satisfy

$|\lambda_s| \le 1$ and every root satisfying $|\lambda_s| \le 1$ have multiplicity not exceeding two, where $\lambda_s, s = 1,2,...,N$ are the roots of $\rho(\lambda) = 0$.

For the block method (6), $\rho(\lambda)$ is obtained as:

$$\rho(\lambda) = \det\left[\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\right] = 0$$

$$= \lambda(\lambda - 1) = 0$$

$$\Rightarrow \lambda_1 = 1 \text{ and } \lambda_2 = 0$$

Hence by Fatunla (1991), the block method (6) is zero-stable since the roots of the characteristic polynomial $\rho(\lambda)$ satisfy the definition above and is consistent since its order $p > 1$ and therefore convergent by Henrici (1962). Similar approach shows that the block methods (9) and (11) are also convergent since they are both zero-stable and consistent.

Region of Absolute Stability

To determine the absolute stability regions of the block methods (6), (9) and (11), we reformulate them as general linear methods of Burrage and Butcher (1980) where a partition of the form

$$\begin{bmatrix} Y \\ Y_{n-i} \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}\begin{bmatrix} hf(y) \\ y_n \end{bmatrix} \qquad (17)$$

was used. The elements A_1 and A_2 are obtained from the coefficients of the collocation points while B_1 and B_2 are obtained from the coefficients of the interpolation points of the methods. The elements of A_1, A_2, B_1 and B_2 are then substituted into a Matlab code for plotting the stability regions of our block methods.

Writing the block method (6) in the form of (17) gives:

$$\begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ \cdots \\ y_{n+2} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \vdots & 0 & 1 \\ \frac{5}{12} & \frac{2}{3} & \frac{-1}{12} & \vdots & 0 & 1 \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & \vdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & \vdots & 0 & 1 \\ \frac{5}{12} & \frac{2}{3} & \frac{-1}{12} & \vdots & 0 & 1 \end{bmatrix}\begin{bmatrix} hf_n \\ hf_{n+1} \\ hf_{n+2} \\ \cdots \\ y_{n+1} \\ y_n \end{bmatrix}$$

where $A_1 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{5}{12} & \frac{2}{3} & \frac{-1}{12} \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} \end{bmatrix} A_2 = \begin{bmatrix} \frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{5}{12} & \frac{2}{3} & \frac{-1}{12} \end{bmatrix} B_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Substituting the values of A_1, A_2, B_1 and B_2 into the MATLAB code for plotting stability regions produces the absolute stability region of the block method (6) as:

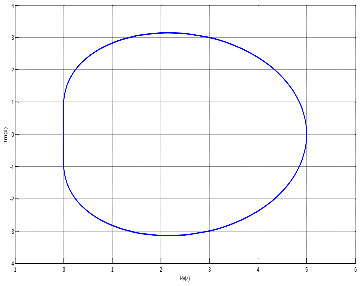

Figure 1: Absolute Stability Region of the Block method (6)

Similarly, the stability regions of the block methods (8) and (10) are obtained as follows:

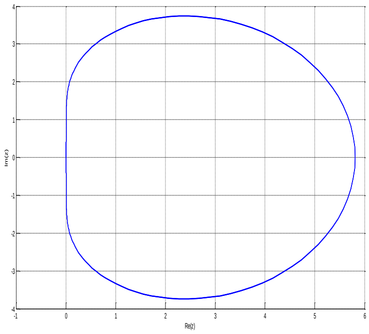

Figure 2: Absolute Stability Region of the Block method (9)

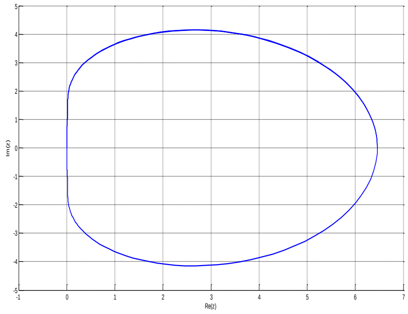

Figure 3: Absolute Stability Region of the Block method (11)

The stability regions of the block methods derived show that the methods are all A – stable. The stability regions consist of the set of points in the complex plane outside the enclosed figures.

NUMERICAL RESULTS

In order to verify the efficiency of the block methods (6),(9) and (11), we present some numerical experiments. The new block methods shall be compared with the analytic solution and the well-known MATLAB ODE solver ode 23s for the problem without analytic solution.

Problem 1.

$$y'_1 = -0.013y_1 - 1000y_1y_3$$
$$y'_2 = -2500y_2y_3$$
$$y'_3 = -0.013y_1 - 1000y_1y_3 - 2500y_2y_3$$
$$y_1(0) = 1, \quad y_2(0) = 1, \quad y_3(0) = 0, \ 0 \le x \le 20, \ h = 0.1$$

Problem 2.

$$y'_1 = 998y_1 + 1998y_2$$
$$y'_2 = -999y_1 - 1999y_2$$
$$y_1(0) = 1, \quad y_2(0) = 1, \quad 0 \le x \le 20, \; h = 0.1$$
$$y_1(x) = 4e^{-x} - 3e^{-1000x}, \; y_2(x) = -2e^{-x} + 3e^{-1000x}$$

Problem 3

$$y'_1 = -1002y_1 + 1000y_2^2$$
$$y'_2 = y_1 - y_2(1 + y_2)$$
$$y_1(0) = 1, \quad y_2(0) = 1, \quad 0 \le x \le 20, \; h = 0.1$$
$$y_1(x) = e^{-2x}, y_2(x) = e^{-x}$$

Problem 4

$$y'_1 = -8y_1 + 7y_2$$
$$y'_2 = 42y_1 - 43y_2$$
$$y_1(0) = 1, \quad y_2(0) = 8, \quad 0 \le x \le 20, \; h = 0.1$$
$$y_1(x) = 2e^{-x} - e^{-50x}, \; y_2(x) = -2e^{-x} + 6e^{-50x}$$

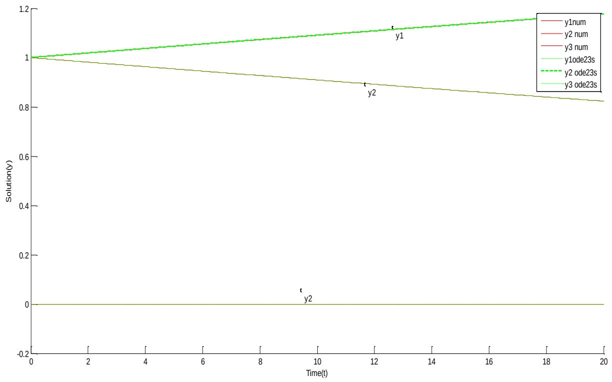

Figure 4: Solution curve to problem 1 using method (6)

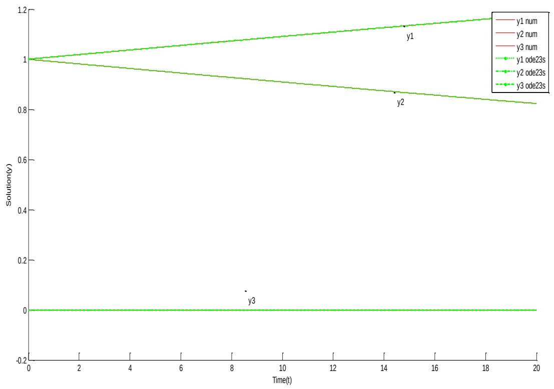

Figure 5: Solution curve to problem 1 using method (9)

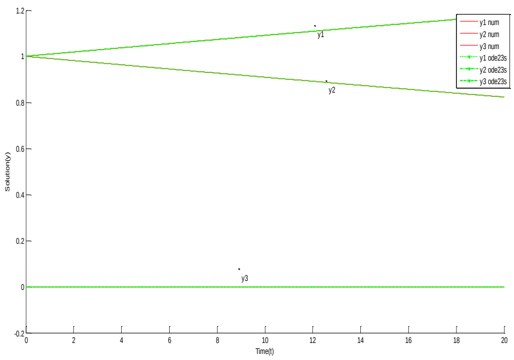

Figure 6: Solution curve to problem 1 using method (11)

Table 2: Absolute errors for problem 2 for the first component

x	Method(6)	Method(9)	Method(11)
2	4.44E-05	2.56E-08	1.74E-07
4	1.16E-05	4.32E-09	4.71E-08
6	2.35E-06	7.02E-10	4.78E-09
8	4.25E-07	1.11E-10	1.73E-09
10	7.19E-08	1.71E-11	2.92E-10
12	1.17E-08	2.61E-12	4.74E-11
14	1.84E-09	3.93E-13	7.49E-12
16	2.85E-10	2.56E-08	1.16E-12
18	4.34E-11	4.32E-09	1.76E-13
20	6.52E-12	7.02E-10	2.65E-14

Table 3: Absolute errors for problem 3 for the first component

x	Method(6)	Method(9)	Method(11)
2	4.49E-02	1.66E-02	2.11E-02
4	1.59E-04	1.86E-04	2.42E-04
6	2.27E-07	1.13E-07	8.19E-07
8	4.29E-08	3.89E-08	3.06E-08
10	1.24E-09	1.20E-09	1.10E-09
12	2.82E-11	2.76E-11	2.65E-11
14	5.79E-13	5.72E-13	5.59E-13
16	1.13E-14	1.13E-14	1.11E-14
18	2.16E-16	2.16E-16	2.14E-16
20	4.07E-18	4.06E-18	1.67E-19

Table 4: Absolute errors for problem 4 for the first component

x	Method(6)	Method(9)	Method(11)
2	2.14E-05	1.29E-06	8.70E-08
4	5.80E-06	3.49E-07	2.36E-08
6	1.18E-06	7.08E-08	4.78E-09
8	2.12E-07	1.28E-08	8.63E-10
10	3.59E-08	2.16E-09	1.46E-10
12	5.83E-09	3.51E-10	2.37E-11
14	9.21E-10	5.54E-11	3.74E-12
16	1.42E-10	8.57E-12	5.79E-13
18	2.17E-11	1.31E-12	8.81E-14
20	3.26E-12	1.96E-13	1.33E-14

CONCLUSION

We have proposed block generalized Simpson-Milne methods for k = 2, 3 and 4 with order three, four and five respectively for the numerical integration of ordinary differential equations. These block methods were found to be zero-stable, consistent and convergent. The numerical examples show that our block methods are highly accurate and compete favourably well with the MATLAB ode solver ode 23s.

REFERENCES

Ajie I. J., Ikhile M. N. O., Onumanyi P. A. (2014). Family of Block Methods Derived from TOM and BDF Pairs for Stiff Ordinary Differential Equations. *American Journal of Mathematics and Statistics* 4(2); 121-130.

Bond, I. and Cash, J., (1979) 'A Block Method for Numerical Integration of Stiff systems of ODEs.' *BIT*, 19(2); 429-447.

Burrage, K. and Butcher, J. C. (1980). Non-linear stability of a general class of differential equation methods. *BIT* 20; 185–203.

Fatunla, S. O. (1991), Block Method for Second order Differential Equations, *International Journal of Computer Mathematics*. 41; 55 – 63.

Milne, W. E. (1953). *Numerical solution of differential equations.* New York; John Wiley and Sons.

Kumleng, G. M., Adee, S. O., Skwame, Y. (2013). Implicit Two Step Adam Moulton Block Method with Two Off-Step Points for Solving Stiff Ordinary Differential Equations. *Journal of Natural sciences Research*. 3(9); 77-81

Lambert, J. D. (1973). *Computational Methods in Ordinary Differential Equations.* New York; John Wiley.

Henrici, P. (1962). *Discrete Variable Method in Ordinary Differential Equations.* New York; John Wiley.

Nasir, N. A. A. M., Ibrahim, Z. B., Suleiman, M. B and Othman, K. I. (2011). Fifth Order Two-point Block Backward Differentiation Formulas for Solving Ordinary Differential Equations. *Applied Mathematical Sciences,* 5(71); 3505 - 3518

Onumanyi, P., Awoyemi, D. O., Jator, S. N. and Sirisena, U. W., (1994) ''New Linear Multistep Methods with Continuous Coefficients for First Order IVPs" *Journal of the Nigerian Mathematical Society*.13.

Rosser, J. B. (1967). A Runge-Kutta for all seasons, *SIAM Rev*. 9; 417 - 452.

Shampine, L. and Watts, H. A., (1972) "A-Stable Block One-Step Methods," *BIT,* 23; 252-266.