# REFORMULATION OF IMPLICIT ONE-STEP LEGENDRE POLYNOMIAL HYBRID BLOCK METHOD IN FORM OF IMPLICIT RUNGE-KUTTA COLLOCATION METHODS FOR THE SOLUTIONS OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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## Abstract

We consider the reformulation of implicit one-step Legendre polynomial hybrid block method in form of implicit runge-kutta collocation methods for the solutions of first-order ordinary differential equations. The methods of uniform accuracy everywhere in the interval of integration of order five (5) were constructed. Some numerical experiments are considered in order to test the method when applied to a variety of initial value problems.

**KEYWORDS:** Hybrid, Implicit, Legendre Polynomial, One-step, Collocation.

## INTRODUCTION

Numerical methods are valuable tools, for finding solutions of ordinary differential equations, since finding an analytic solution is often very difficult or impossible. The first numerical method for ordinary differential equations is the famous Euler method introduced in 1760's and republished in his collected works in 1913.

In this paper, we consider the reformulation of implicit one-step Legendre polynomial block hybrid – method into implicit Runge-kutta collocation methods, due to their excellent stability and stiffly accurate characteristic properties for the direct integration of initial value problem, possibly stiff, of the form:

$$y'(x) = f(x, y(x)),\ y(x_\circ) = y_\circ, \quad (x_\circ \le x \le b). \tag{1}$$

Here the unknown function $y$ is a mapping $[x_\circ, b] \to \Re^N$, the right-hand side function $f$ is $[x_\circ, b] \times \Re^N \to \Re^N$ and the initial vector $y(x_\circ)$ is given in $\Re^N$. This method evaluates the driving function of (1) once in each step and uses an approximated solution from the previous step to up-date a solution. Early extensions of the method is the well-known and the most commonly used Runge-kutta method

(Multistage), because the Runge-kutta uses a result given at the one or more off-step points (Chollom and Jackiewicz, 2003; Euler, 1913).

Different methods have been proposed for the solution of (1) ranging from predictor-corrector methods to hybrid methods. One-step methods have always been regarded as expensive because of their multistage structure (Burrage and Butcher 2001). The implementation costs for implicit Runge-kutta methods present obstacle to finding cheap implementation because of the structure of the coefficient matrix A in Butcher's array, which has a pair of complex conjugate eigen-values. For both explicit and implicit Runge-kutta methods it is very difficult to estimate errors for variable order $p$ (Burrage and Butcher 2001).

Hybrid methods have advantage of incorporating function evaluation at off-step points which affords the opportunity of circumventing the Dahlquist zero stability barrier and it is actually possible to obtain convergent k-step methods with order 2k+1 up to k=9, (Butcher, 2003). The method is also useful in reducing the step number of a method and still remains zero stable (Adesanya *et al.* 2013).

## MATERIALS AND METHODS

### Derivation of the Implicit Runge-Kutta Collocation Method

We consider a k-step collocation method (Onumanyi, *et al.* 1994) which was a generalization of the form presented by ( Lei, and Nosett, 1989):

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + h\sum_{j=0}^{s-1} \beta_j(x) f_{n+j} \qquad (2)$$

Where $t$ denote the number of interpolation points $x_j$ , $j = 0,1,2,\cdots,t-1$ and $s$ denotes the distinct collocation points $\bar{x}_j \in [x_\circ, b],\ j = 0,1,2,\cdots,s-1$, belonging to the given interval. The step size h can be a variable, it is assumed in this paper as a constant for simplicity, with the given mesh $x_n : x_n = x_\circ + nh,\quad n = 0,1,2,\cdots,N$,

where $h = x_{n+1} - x_n$ , $N = \dfrac{(b-a)}{h}$ , and a set of equally spaced points on the integration interval given by $x_\circ < x_1 < \cdots < x_{n+1} = b$ . Also we assumed that (1) has a unique solution and $\alpha_j(x)\ and\ h\beta_j(x)$ in (2) are to be represented by the polynomials:

$$\alpha_j(x) = \sum_{i=0}^{t+s-1} \alpha_{j,i+1} x^i,\ \ h\beta_j(x) = \sum_{i=0}^{t+s-1} h\beta_{j,i+1} x^i , \qquad (3)$$

With constant coefficients $\alpha_{j,i+1}$ and $\beta_{j,i+1}$ to be determined. Expanding $y(x)$ in (2) using Taylor series method of expansion about $x$ and collect powers in h to obtain the methods.
Inserting (3) into (2) we have:

$$\begin{aligned} y(x) &= \sum_{j=0}^{t-1}\sum_{i=0}^{t+s-1}\alpha_{j,i+1}x^i y_{n+j} + \sum_{j=0}^{s-1}\sum_{i=0}^{t+s-1}h\beta_{j,i+1}x^i f_{n+j} \\ &= \sum_{i=0}^{t+s-1}\left\{\sum_{j=0}^{t-1}\alpha_{j,i+1}y_{n+j} + \sum_{j=0}^{s-1}h\beta_{j,i+1}f_{n+j}\right\}x^i \end{aligned} \tag{4}$$

Written as

$$a_i = \sum_{j=0}^{t-1}\alpha_{j,i+1}y_{n+j} + \sum_{j=0}^{s-1}h\beta_{j,i+1}f_{n+j}$$

Such that (4) reduces to

$$y(x) = \sum_{i=0}^{t+s-1}a_i x^i \tag{5}$$

which can now be expressed in the form

$$y(x) = \left\{\sum_{j=0}^{t-1}\alpha_{j,t+s-1}y_{n+j} + \sum_{j=0}^{s-1}h\beta_{j,t+s-1}f_{n+j}\right\}(1, x, x^2, \cdots, x^{t+s-1})^T.$$

Thus, we can express equation (5) explicitly as follows:

$$y(x) = (y_n, \cdots, y_{n+t-1}, f_n, \cdots, f_{n+s-1})C^T(1, x, x^2, \cdots, x^{t+s-1})^T \tag{6}$$

where

$$C = \begin{pmatrix} c_{1,0} & \cdots & c_{1,t} & c_{1,t+1} & \dots & c_{1,t+s} \\ c_{2,0} & \cdots & c_{2,t} & c_{2,t+1} & \dots & c_{2,t+s} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{t,1} & \cdots & c_{t,t} & c_{t,t+1} & \dots & c_{t,t+s} \\ c_{t+1,1} & \cdots & c_{t+1,t} & c_{t+1,t+1} & \dots & c_{t+1,t+s} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{t+s,1} & \cdots & c_{t+s,t} & c_{t+s,t+1} & \dots & c_{t+s,t+s} \end{pmatrix} = D^{-1} \tag{7}$$

and

$$D=\begin{pmatrix} 1 & x_n & x_n^2 & \cdots x^i{}_n & \cdots & x_n^{t+s-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots x^t{}_{n+1} & \ldots & x_{n+1}^{t+s-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \ddots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots x^t{}_{n+t-1} & \ldots & x_{n+t-1}^{t+s-1} \\ 0 & 1 & 2\bar{x}_0 \cdots & tx_i^{(i-1)} & \cdots & (t+s-1)\bar{x}_{is}^{t+s-2} \\ \vdots & \vdots & \vdots & \ddots \vdots & \ddots & \vdots \\ 0 & 1 & 2\bar{x}_s & \cdots t\bar{x}_s^{(t-1)} & \cdots & (t+s-1)\bar{x}_{n+s}^{t+s-2} \end{pmatrix} \tag{8}$$

are matrices of dimensions (t+s)x(t+s). We call D the multistep collocation and interpolation matrix which has a very simple structure. It is similar to vandermonde matrix, consisting of distinct elements, non-singular, and of dimension (s+t)x(s+t). This matrix affects the efficiency, accuracy and stability properties of (4) the choice $C=D^{-1}$ leads to the determination of the constant coefficients $\alpha_{j,i+1}$ and $\beta_{j,i+1}$. It was shown in Jain *et al.*, (2007), Butcher, (2008) and Chollom and Donald, (2009) that the method (2) is convergent with order $p=t+s-1$. We now examine in more detail how the constant coefficients $\alpha_{j,i+1}$ and $\beta_{j,i+1}$ of equation (2) are obtained for the reformulation of implicit one-step Legendre polynomial block hybrid method into implicit Runge-kutta methods.

Therefore, by careful selection of the interpolation and collocation points inside the interval $[x_\circ, b]$, leads to a single continuous finite difference method whose members are of uniform accuracies (Butcher, 2003; Butcher, 2005). The matrix D of equation (8) takes the form:

$$D=\begin{pmatrix} 7 & 11x_n & -18x_n^2 & -50x_n^3 & 35x_n^4 & 63x_n^5 \\ 0 & 11 & -36x_n^2 & -150x_n^3 & 140x_n^4 & 315x_n^5 \\ 0 & 11 & -36x_{n+u}^2 & -150x_{n+u}^3 & 140x_{n+u}^4 & 315x_{n+u}^5 \\ 0 & 11 & -36x_{n+w}^2 & -150x_{n+w}^3 & 140x_{n+w}^4 & 315x_{n+w}^5 \\ 0 & 11 & -36x_{n+v}^2 & -150x_{n+v}^3 & 140x_{n+v}^4 & 315x_{n+v}^5 \\ 0 & 11 & -36x_{n+1}^2 & -150x_{n+1}^3 & 140x_{n+1}^4 & 315x_{n+1}^5 \end{pmatrix} \tag{9}$$

Where u ,w and v are zeros of $l_m(x)=0$, Legendre polynomials of degree m=3 (Villadsen and Michelsen, 1987).

$$\bar{x}_{\circ} = x_{n+u} \;, u = \frac{1}{4} \;, \; \bar{x}_1 = x_{n+w} \;, w = \frac{1}{2} \; and \; \bar{x}_2 = x_{n+v} \;, \; v = \frac{3}{4} \qquad (10)$$

Which are valued in the interval $[x_{\circ}, b]$. Inverting the matrix D in equation (9) once, using computer algebra, for example, Maple or MatLab software package, we obtain the continuous scheme as

$$y(x) = \alpha_0(x) y_n + \left[\beta_0(x) f_n + \beta_u(x) f_{n+u} + \beta_w(x) f_{n+w} + \beta_v(x) f_{n+v} + \beta_1(x) f_{n+1}\right] \qquad (11)$$

Where

$$\alpha_0(x) = 1$$

$$\beta_0(x) = -\frac{1}{90}\left[192x^5 - 600x^4 + 700x^3 - 375x^2 + 90x\right]$$

$$\beta_u(x) = \frac{1}{45}\left[-384x^5 + 1080x^4 - 1040x^3 + 360x^2\right]$$

$$\beta_w(x) = -\frac{1}{15}\left[192x^5 - 480x^4 + 380x^3 - 90x^2\right]$$

$$\beta_v(x) = \frac{1}{45}\left[-384x^5 + 840x^4 - 560x^3 + 120x^2\right]$$

$$\beta_1(x) = -\frac{1}{90}\left[192x^5 - 360x^4 + 220x^3 - 45x^2\right]$$

We evaluate $y(x)$ in (11) at the following point $x = x_{n+1}, x_{n+u}, x_{n+w} \; and \; x_{n+v}$, we obtain the following 4-block hybrid scheme with uniformly accurate order five:

$$y_{n+1} = y_n + \frac{h}{90}\left[7 f_n + 32 f_{n+u} + 12 f_{n+w} + 32 f_{n+v} + 7 f_{n+1}\right] \qquad (12a)$$

$$order \; p = 6 \,, \quad C_7 = -\frac{1}{1935360}$$

$$y_{n+u} = y_n + \frac{h}{2880}\left[251 f_n + 646 f_{n+u} - 264 f_{n+w} + 106 f_{n+v} - 19 f_{n+1}\right] \qquad (12b)$$

$$order \; p = 5 \,, \quad C_6 = \frac{3}{655360}$$

$$y_{n+w} = y_n + \frac{h}{360}\left[29 f_n + 124 f_{n+u} + 24 f_{n+w} + 4 f_{n+v} - f_{n+1}\right] \qquad (12c)$$

$$order \; p = 5 \,, \quad C_6 = \frac{1}{368640}$$

$$y_{n+v} = y_n + \frac{h}{320}\left[27f_n + 102f_{n+u} + 72f_{n+w} + 42f_{n+v} - 3f_{n+1}\right] \tag{12d}$$
$$order\ p = 5, \quad C_6 = \frac{3}{655360}$$

We converted the implicit one-step Legendre polynomial block hybrid scheme above to implicit Runge-kutta collocation method, written as:

$$y_n = y_{n-1} + h\left(\frac{7}{90}\right)F_1 + h\left(\frac{16}{45}\right)F_2 + h\left(\frac{2}{15}\right)F_3 + h\left(\frac{16}{45}\right)F_4 + h\left(\frac{7}{90}\right)F_5 \tag{13}$$

The stage values at the nth step are computed as:

$$Y_1 = y_{n-1}$$
$$Y_2 = y_{n-1} + h\left(\frac{251}{2880}\right)F_1 + h\left(\frac{323}{1440}\right)F_2 + h\left(\frac{11}{120}\right)F_3 + h\left(\frac{53}{1440}\right)F_4 - h\left(\frac{19}{2880}\right)F_5$$
$$Y_3 = y_{n-1} + h\left(\frac{29}{360}\right)F_1 + h\left(\frac{31}{90}\right)F_2 + h\left(\frac{1}{15}\right)F_3 + h\left(\frac{1}{90}\right)F_4 - h\left(\frac{1}{360}\right)F_5$$
$$Y_4 = y_{n-1} + h\left(\frac{27}{320}\right)F_1 + h\left(\frac{51}{160}\right)F_2 + h\left(\frac{9}{40}\right)F_3 + h\left(\frac{21}{160}\right)F_4 + h\left(\frac{3}{320}\right)F_5$$

With the stage derivatives as follows:

$$F_1 = f\left(x_{n-1} + h(0), Y_1\right)$$
$$F_2 = f\left(x_{n-1} + h\left(\frac{1}{4}\right), Y_2\right)$$
$$F_3 = f\left(x_{n-1} + h\left(\frac{1}{2}\right), Y_3\right)$$
$$F_4 = f\left(x_{n-1} + h\left(\frac{3}{4}\right), Y_4\right)$$
$$F_5 = f\left(x_{n-1} + h(1), Y_5\right)$$

The implicit Runge-kutta collocation method has order P=5, the Butcher tableau that defines the method as in Chollom and Jackiewicz, (2003) and Yakubu (2003), takes the form:

$$
\begin{array}{c|ccccc}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\frac{1}{4} & \frac{251}{2880} & \frac{323}{1440} & \frac{11}{120} & \frac{53}{1440} & -\frac{19}{2880} \\
\frac{1}{2} & \frac{29}{360} & \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & -\frac{1}{360} \\
\frac{3}{4} & \frac{27}{320} & \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & -\frac{3}{320} \\
\mathbf{1} & \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \\
\hline
 & \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90}
\end{array}
$$

**Analysis of Basic Properties of the Method**

The convergence and stability properties of the methods are discussed by reformulation of the block method as general linear methods (Butcher, 2008; Burrage and Butcher, 2001). Hence, we use the notation introduced by Butcher, (2008), where a general linear method is represented by a partitioned $(s+r)\times(s+r)$ matrix.

$$M=\begin{bmatrix} A & U \\ B & V \end{bmatrix} \qquad (14)$$

Hence it takes the form

$$\begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \vdots \\ Y_s^{[n]} \\ ---- \\ y_1^{[n]} \\ \vdots \\ y_s^{[n]} \end{bmatrix} = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \begin{bmatrix} hf(Y_1)^{[n]} \\ hf(Y_2)^{[n]} \\ \vdots \\ hf(Y_s)^{[n]} \\ ---- \\ y_1^{[n-1]} \\ \vdots \\ y_s^{[n-1]} \end{bmatrix}, \; \text{n} = 1,2,\cdots,\text{N}, \qquad (15)$$

where

$$A=\begin{bmatrix}0 & 0\\ A & B\end{bmatrix},\quad U=\begin{bmatrix}1 & 0 & 0\\ 0 & \psi & c-\psi\end{bmatrix},\quad B=\begin{bmatrix}A & B\\ 0 & 0\\ V^T & \omega^T\end{bmatrix}, and\quad V=\begin{bmatrix}I & \psi & c-\psi\\ 0 & 0 & I\\ 0 & 0 & I-\theta\end{bmatrix}$$

where **r** denotes quantities as output from each step and input to the next step and **s** denotes stage values used in the computation of the step $Y_1, Y_2, \ldots, Y_s$. The coefficients of these matrices, A, U, B and V indicate the relationship between the various numerical quantities that arise in the computation of stability regions. The elements of the matrices A, U, B and V are substituted into the stability matrix. Applying (15) to the test equation $y' = \lambda^2 y,\ x \geq 0\ and\ \lambda \in \not\subset$ leads to the recurrent equation:

$$y^{[n-1]} = M(z)y^{[n]},\ n = 1,2,3,\ldots, N-1,\ z = \lambda h,$$

where the stability matrix

$$M(z) = V + zB(1 - zA)^{-1}U$$

and the stability polynomial of the method can easily be obtained as follows:

$$\rho(\eta, z) = \det(\eta I - M(z)).$$

The absolute stability region of the method is defined as

$$\Re = x \in \not\subset : \rho(\eta, z) = 1 \Rightarrow |\eta| \leq 1.$$

Computing the stability functions gives the stability polynomial of the method as:

$$\bar{h}(\omega) = -h^4\left(\frac{1}{1280}\omega^3 - \frac{1}{1280}\omega^4\right) - h^3\left(\frac{5}{384}\omega^4 + \frac{5}{384}\omega^3\right) - h^2\left(\frac{7}{64}\omega^3 - \frac{7}{64}\omega^4\right) - h\left(\frac{1}{2}\omega^4 + \frac{1}{2}\omega^3\right) + \omega^4 - \omega^3 \qquad (16)$$

The stability polynomial (16) is plotted to produce the required graph of the absolute stability region of the method as displayed in the figure 1.

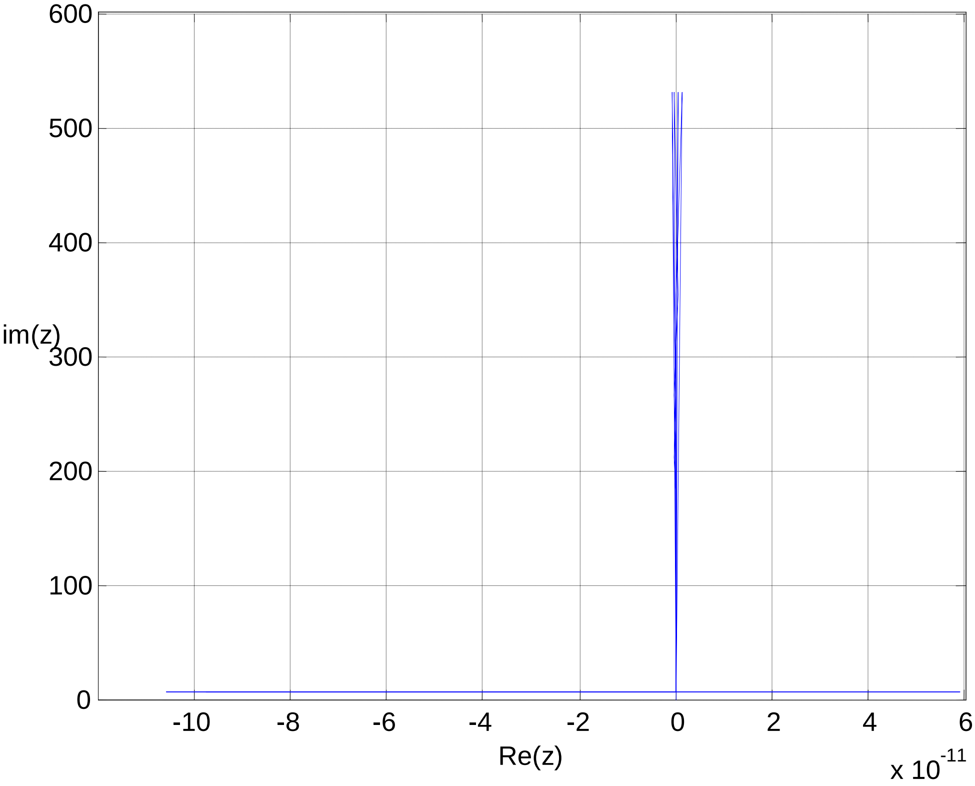


**Figure.1:** Stability region of the implicit one-step Runge-kutta collocation method.

**Numerical Experiments**

The new method shall be tested on some set of stiff differential equations and compared the obtained results side by side in tables. The numerical results were obtained using MATLAB software.

***Problem 1***

A certain radioactive substance is known to decay at the rate proportional to the amount present. A block of this substance having a mass of 100g originally is observed. After 40hours, its mass reduced to 90g. Test for the consistency of the method on this problem for $t \in [0,1]$.

This stiff problem is modeled by the differential equation:

$$\frac{dN}{dt} = \alpha N, \alpha = -0.0026, N(0) = 100, t \in [0,1] \qquad (17\ )$$

Where N represents the mass of the substance at any time $t$ and $\alpha$ is a constant which specifies the rate at which this particular substance decays. The theoretical solution to (17 ) is given by;

$$N(t) = 100e^{-0.0026t} \tag{18}$$

This problem was also solved by Sunday (2011), where they proposed on Adomian decomposition method for numerical solution of ODEs arising from the natural laws of growth and decay. Result to this problem is shown in table 1.

**Table 1:** comparison of the exact solution with the approximation/computed solution of problem (1)

| t | Exact result | Computed result | Absolute Errors | Time/s |
|---|---|---|---|---|
| 0.10 | 99.9740033797 | 99.9740033797 | 0.000E+00 | 0.0191 |
| 0.20 | 99.9480135177 | 99.9480135177 | 0.000E+00 | 0.0232 |
| 0.30 | 99.9220304121 | 99.9220304121 | 0.000E+00 | 0.0272 |
| 0.40 | 99.8960540613 | 99.8960540613 | 0.000E+00 | 0.0314 |
| 0.50 | 99.8700844634 | 99.8700844634 | 0.000E+00 | 0.0349 |
| 0.60 | 99.8441216168 | 99.8441216168 | 0.000E+00 | 0.0389 |
| 0.70 | 99.8181655196 | 99.8181655196 | 0.000E+00 | 0.0435 |
| 0.80 | 99.7922161701 | 99.7922161701 | 0.000E+00 | 0.0475 |
| 0.90 | 99.7662735666 | 99.7662735666 | 0.000E+00 | 0.0521 |
| 1.00 | 99.7403377073 | 99.7403377073 | 0.000E+00 | 0.0562 |

***Problem 2***

Consider the stiff initial value problem

$$y' = 5y,\ y_0 = 1 \tag{19}$$

With the exact solution

$$y(x) = e^{5x}\ . \tag{20}$$

**Table 2:** Comparison of exact solution with approximate solution of problem (2), h=0.01

| x | Exact result | Computed result | Absolute Errors | Time/s |
|---|---|---|---|---|
| 0.00 | 1.000000000 | 1.000000000 | 0.000E+00 | 0.0822 |
| 0.01 | 1.051271096 | 1.051271096 | 0.000E+00 | 0.1090 |
| 0.02 | 1.105170918 | 1.105170917 | 1.000E-09 | 0.1098 |
| 0.03 | 1.161834243 | 1.161834244 | 1.000E-09 | 0.1106 |
| 0.04 | 1.221402758 | 1.221402758 | 0.000E+00 | 0.1124 |
| 0.05 | 1.284025417 | 1.284025418 | 1.000E-09 | 0.1132 |
| 0.06 | 1.349858808 | 1.349858808 | 0.000E+00 | 0.1141 |
| 0.07 | 1.419067549 | 1.419067551 | 2.000E-09 | 0.1151 |
| 0.08 | 1.491824698 | 1.491824700 | 2.000E-09 | 0.1162 |
| 0.09 | 1.568312185 | 1.568312188 | 3.000E-09 | 0.1171 |
| 0.10 | 1.648721271 | 1.648721273 | 2.000E-09 | 0.1183 |

## CONCLUSION

In this paper we have studied the class of one-step implicit Runge-kutta collocation methods suitable for the approximate numerical integration of first order ordinary differential equations. We obtained uniformly accurate order five method at the step points as well as at some selected off-step points. In this way acceptable stability for stiff problems is obtained as for the Runge-kutta methods. All the derived methods obtained through this approach performed very well in stiff and initial value problem of first order ordinary differential equations.

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