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Modelling the Effect of Public Campaign, Case Detection, Vaccine, Quarantine and Sanitarium on the Transmission of Tuberculosis Infections

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Abstract
In this paper, a mathematical model for transmission of TB was developed. In developing the model, the human population was positioned into; susceptible, exposed, vaccinated, quarantine, sanitarium, infected and recovered compartments. A basic model consisting of seven differential equations was developed where the existence and uniqueness of solution to the differential equations was proofed. Qualitative analysis of model by proving the positivity of solution and establishment of the invariant region of the solution. The reproduction number R_0 was also obtained, and local stability analysis of the solution was carried out and we established that ; disease free equilibrium point was locally asymptotically stable (LAS). We also carried out sensitivity analysis of the parameter. Where the parameters (ρ , d_2 , α , β , ω , Λ , p , μ) were shown to have been an inverse proportional relationship with R_0. The numerical experiment also suggest that case detection, quarantine, sanitarium and public campaign plays a vital role in curtailing the spread of TB among populace.

Keywords: Tuberculosis, Sensitivity analysis, reproduction number, management, complication.

Introduction

Tuberculosis (TB) is a chronic infectious disease caused by mycobacterium (*Tubercle bacillus*). The disease spreads from one individual to another through the air. Infected persons release droplets of Mycobacterium tuberculosis bacteria into the air by coughing, sneezing or spitting mucus containing the bacteria onto surfaces. This droplets or mucus contain large number of small respiratory droplets nuclei that evaporate and disperse into the air almost instantly. This nuclei implant themselves in the lungs when inhaled. In most cases, a competent immune system limits the multiplication of the tuberculosis bacilli, although some bacilli remain dormant but viable, rendering a condition known as latent TB infection (LTBI), (CDC, 2000).

Worldwide, 8.6 million people fell ill due to TB, of which 1.3 million people die annually WHO (2013). In Africa, the TB incidences per 100,000 population is 262 while the prevalence is 293 as per WHO (2009). At global level, TB is the seventh most important cause of global premature mortality and disability and it is projected to remain among the ten leading causes of disease burden even in the year 2020 (Nyerere *et al*; 2014).

The use of mathematical modeling in theory and practice of disease management and control have increase due to the fact that, the approach helps in figuring out control strategies and making decisions that are of significant importance on the control of the disease (Nyerere, *et al.*, 2014).

Mathematical models have played a key role in understanding and formulation of TB control strategies, which can be used in establishment of interim goals for intervention programs: for instance; Waaler *et al.*, (1962) developed the mathematical model for the dynamics of susceptible, exposed and infected individuals; (Castillo & Feng *et al.*, 1997); Mathematical model for multiple strains and variables latent period by Feng *et al.*, (2002); two vaccine

Mathematical model incorporating pre-exposure and post-exposure by Lietman & Blower (2000). Also Daniel and Andrei (2007) developed a SEIJT (Susceptible-Exposed-Undetected-Infected-Detected-Infected-Treated) model on the effect of Directed Observation Therapy Strategy (DOTS) in Nigeria. Their results showed that, if the fraction of detected infectious individuals exceeded a critical value, there exists a globally stable disease free equilibrium. However, if this critical detection level is not reached, the disease free equilibrium will be unstable even with the very high probability successful treatment under DOTS. The above mentioned models were based on the SEIR model architecture.

Standard anti-TB drugs have been used for decades but resistance to the medicines is growing. Knowing that the resistance is still growing, there is the need for more comprehensive and renewed concern over the disease (TB). And a good understanding of effective treatment and controlled strategies in different regions of the world is still needed. (CDC, 2000). A person with LTBI has an estimated 10% life-time risk of developing active TB disease. However, certain persons, like children under 4 years of age, persons who have a weakened immune system due to conditions such as malnutrition, HIV/AIDS, diabetes, or certain cancers and those recently infected persons with mycobacterium tuberculosis have a much greater risk of developing active TB. HIV co-infection is the strongest known risk factor for developing active TB disease. Studies suggest that being co-infected with HIV and TB places people at a 7-10% per year risk of developing active TB, (CDC, 2000). Also Outbreaks of TB is common from prisons, nursing homes, residential Centres, urban homeless shelters, aircraft, school and bars due to crowded living conditions with prolonged close exposure to an infectious person.

TB can be managed and control through the following: Public campaign to create awareness on preventive hygiene and avoiding crowding in a localized environment, vaccination, quarantine and sanitarium for preventing the progression from latent infection to active TB through, drug treatment. The above management and control approach may include; Direct Observation Therapy Strategy (DOTS), routine – collected data, short course chemotherapy on TB, early identification, screening and isolation of infectious TB patients; effective engineering controls (*environmental controls such as general ventilation, high – efficiency particulate air [HEPA*) filters, or Ultraviolet Germicidal Irradiation [UVGGI]); the adoption of appropriate respiratory protection (*surgical masks and particulate respirators such as HEPA Masks*); health – care worker TB training, education, counseling.

Materials and methods

The total population is divided into seven compartments (*depending on the epidemiological status of individuals*) as follows: susceptible $S(t)$, Exposed $E(t)$, Infected $I(t)$, Quarantined $Q(t)$, Sanitarium $J(t)$, Vaccinated $V(t)$ and Recovered $R(t)$ compartments respectively. We refer to such model as SEIQJVR.

We make the following assumptions for our model:

(i) All exposed individuals and immigrants suspected of TB case are quarantined.
(ii) Treatment of infected individuals occur in the sanitarium.
(iii) Quarantined individuals who are tested positive will be taken to sanitarium for immediate treatment.
(iv) It is only infected individuals can transmit the disease
(v) Screening is done to the exposed individual and immigrants to detect infected individuals.
(vi) Natural death occurs across the compartments.
(vii) All exposed individuals and the immigrants that tested negative are going to the quarantine.
(viii) An immigrant can fall into infected and quarantine classes
(ix) There is screening of the immigrants at the border.
(x) Recruitment rate to be constant.

(xi) Those on vaccine become exposed on expiration of the vaccine efficacy.

Table 1: Definitions of Variables

Variables	Definitions
S(t)	The number of susceptible individuals at time, t
E(t)	The number of exposed individuals at time, t
I(t)	The number of infected individuals at time, t
Q(t)	The number of individuals suspected with the symptoms of the disease at time, t
J(t)	The number of individuals in a sanitarium at time, t
$V(t)$	The number of vaccinated individuals at time t.
$R(t)$	The number of recovered/treated individuals at time, t
N(t)	Total population at time, t

Table 2: Definitions of parameters

Parameters	Definitions
Λ	Birth Rate
μ	Natural mortality rate
ρ	Proportion of vaccination at birth
$1-p$	Proportion of those not vaccinated at birth
ϕ_1	Rate of Immigrants tested to be negative
d_2	TB induced death rate for individuals in the sanitarium
d_1	TB induced death rate for infected individuals
ϕ_2	Rate of Immigrants tested to be positive
ω	Proportion of quarantined individual that are infected go for treatment
$1-\omega$	Proportion of quarantine individuals that are not infected
q	The rate at which susceptible individuals are vaccinated
α	The rate at which infected individuals move to sanitarium
β	per capita transmission rate from susceptible to exposed class
θ	The rate at which new born are vaccinated
r_1	The recovery rate of individuals in the sanitarium
π	The efficacy of vaccine
λ	The force of infection
p	The rate of movement from expose to infected class
γ	Proportion of exposed individuals quarantined
$1-\gamma$	Proportion of exposed individuals that are not quarantined
η	probability of acquiring TB infections per contact with one infectious individual.

Model description and formulation

The total population is divided into seven compartments (*depending on the epidemiological status of individuals*) as follows: susceptible $S(t)$, Exposed $E(t)$, Infected $I(t)$, Quarantined $Q(t)$, Sanitarium $J(t)$, Vaccinated $V(t)$ and Recovered $R(t)$ compartments respectively. We refer to such model as SEIQJVR. The dynamic of the model is as follows: the recruitment into the vaccinated compartment by vaccinating a proportion $\rho\Lambda$ at birth and is reduce by θv due to lose of vaccine efficacy and by natural death μ v. the recruitment into the susceptible compartment is by proportion(**1**-ρ) Λ birth rate that were not vaccinated and birth. The susceptible population reduces by natural death at the rate μ s and those that progresses to the exposed compartment by λ s. the infected class increases by immigration at the rate ϕ_2 and by those confirm infected at the rate p $(1-\gamma)$ E and reduced by $(\mu + d_1)$ I due to natural and TB induced deaths, also reduces by α I due to proposion that goes for treatment. Quarantine compartment increases by immigrants suspected ϕ_1 and those suspected in exposed class at the rate of γE and reduces by natural death μ Q and those that goes for treatment at the rate ω Q and to the susceptible class by $(1-\omega)$ Q. the sanitarium compartment increases by proportion of those that goes for treatment by α I and ω Q and reduces by r_1 J, due to effective treatment, also reduce by $(\mu + d_2)$J due to natural death and TB induce death. The recovered compartment increases by those that recovered from the disease at the rate r_1 J and reduces by natural death μ R.

From our assumptions and the model description above we have the model flow diagram below

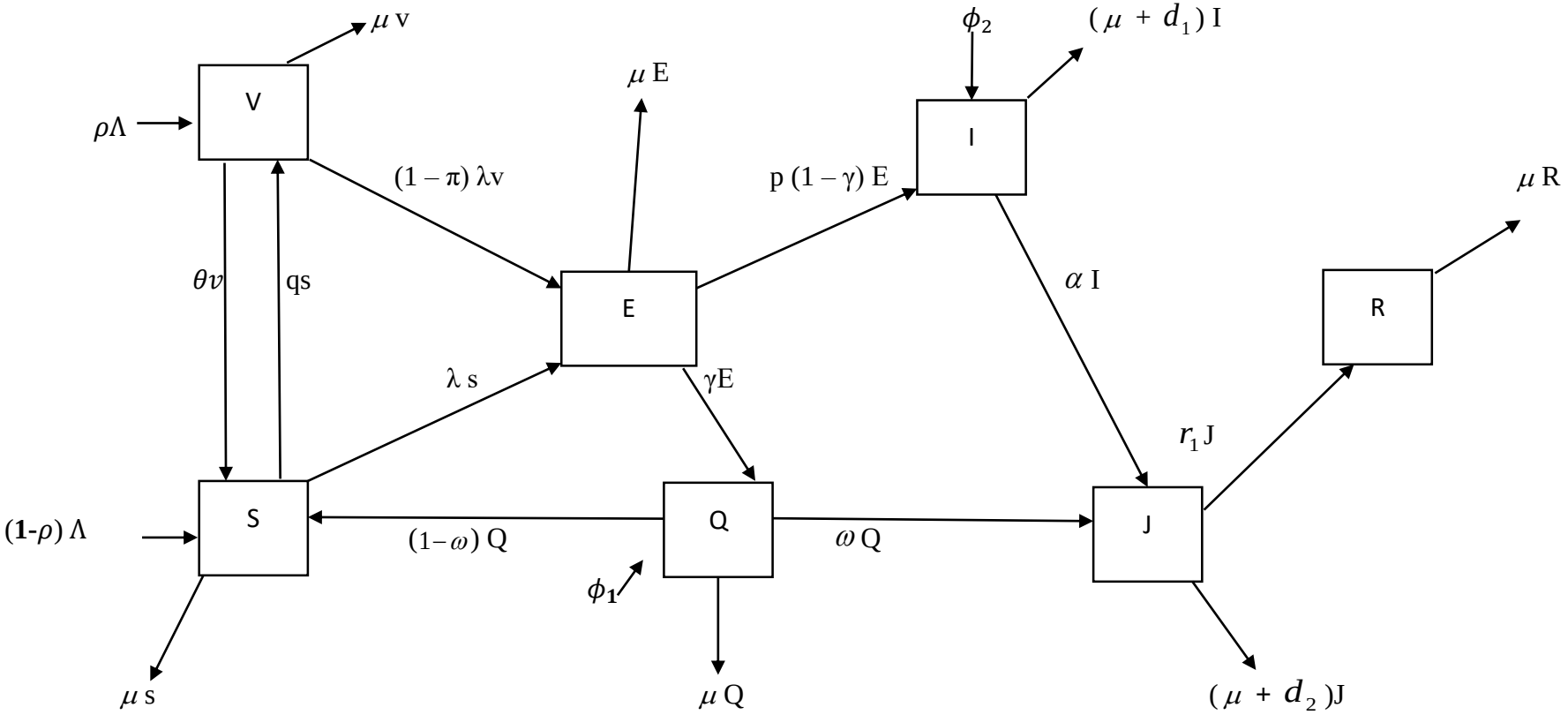

Figure 1: A general transfer diagram for the modified model (SEIQJVR)

Model equations

Based on the assumptions and the flow diagram in figure 2, we develop the following set of ODEs for our modified model (SEIQJVR).

$$\begin{aligned}
\frac{dS}{dt} &= (1-\rho)\Lambda + \theta V + (1-\omega)Q - qS - \mu S - \lambda S \\
\frac{dE}{dt} &= (1-\pi)\lambda V + \lambda S - p(1-\gamma)E - \gamma E - \mu E \\
\frac{dI}{dt} &= p(1-\gamma)E + \phi_2 - \alpha I - (\mu + d_1)I \\
\frac{dQ}{dt} &= \gamma E - (1-\omega)Q - \omega Q - \mu Q + \phi_1 \\
\frac{dJ}{dt} &= \alpha I + \omega Q - r_1 J - (\mu + d_2)J \\
\frac{dV}{dt} &= \rho\Lambda + qS - \theta V - (1-\pi)\lambda V - \mu V \\
\frac{dR}{dt} &= r_1 J - \mu R \\
N(t) &= S(t) + E(t) + I(t) + Q(t) + J(t) + V(t) + R(t)
\end{aligned} \qquad (1)$$

Where $\lambda = \frac{\beta\,(I + \eta J)}{N}$

Results and Discussion

Existence and Uniqueness

Lemma 1: if $f(t, y)$ has continuous partial derivative $\frac{\partial f_i}{\partial y_i}$ for $i = 1, 2, \cdots, n$ on a bounded convex domain R, then it satisfies a Lipchitz condition in R $\|f(t, y) - f(t, y_{n-1})\| \le k\|y_n - y_{n-1}\|, i = 1, 2, 3, \cdots$

let

$$\left.\begin{aligned}
f(t,y_1) &= (1-\rho)\Lambda + \theta v + (1-\omega)Q - qs - \mu s - \lambda s \\
f(t,y_2) &= (1-\pi)\lambda v + \lambda s - p(1-\gamma)E - \gamma E - \mu E \\
f(t,y_3) &= p(1-\gamma)E + \phi_2 - \alpha I - (\mu + d_1)I \\
f(t,y_4) &= \gamma E - (1-\omega)Q - \omega Q - \mu Q + \phi_1 \\
f(t,y_5) &= \alpha I + \omega Q - r_1 J - (\mu + d_2)J \\
f(t,y_6) &= +\rho\Lambda + qs - \theta v - (1-\pi)\lambda v - \mu v \\
f(t,y_7) &= r_1 J - \mu R
\end{aligned}\right\} \quad (2)$$

To show that $\frac{\partial f_i}{\partial y_j}, i, j = 1, 2, 3, \cdots, 7$ are continuous. We consider the partial derivatives of equation (2)

$$\frac{\partial f_1}{\partial S} = -(q+\mu+\lambda) < \infty, \frac{\partial f_1}{\partial E} = 0 < \infty, \frac{\partial f_1}{\partial I} = 0 < \infty, \frac{\partial f_1}{\partial Q} = 0 < \infty, \frac{\partial f_1}{\partial J} = 0 < \infty, \frac{\partial f_1}{\partial V} = 0 < \infty, \frac{\partial f_1}{\partial R} = 0 < \infty$$

$$\frac{\partial f_2}{\partial S} = \lambda < \infty, \frac{\partial f_2}{\partial E} = -\rho(1-\gamma) - (\lambda+\mu) < \infty, \frac{\partial f_2}{\partial I} = 0 < \infty, \frac{\partial f_2}{\partial Q} = 0 < \infty, \frac{\partial f_2}{\partial J} = 0 < \infty, \frac{\partial f_2}{\partial V} = 0 < \infty, \frac{\partial f_2}{\partial R} = 0 < \infty$$

$$\frac{\partial f_3}{\partial S} = 0 < \infty, \frac{\partial f_3}{\partial E} = \rho(1-\gamma) < \infty, \frac{\partial f_3}{\partial I} = -(\alpha + r_2 + \mu + d_1) < \infty, \frac{\partial f_3}{\partial Q} = 0 < \infty, \frac{\partial f_3}{\partial J} = 0 < \infty, \frac{\partial f_3}{\partial V} = 0 < \infty, \frac{\partial f_3}{\partial R} = 0 < \infty$$

$$\frac{\partial f_4}{\partial S} = 0 < \infty, \frac{\partial f_4}{\partial E} = \gamma < \infty, \frac{\partial f_4}{\partial I} = 0 < \infty, \frac{\partial f_4}{\partial Q} = -(1+\mu) < \infty, \frac{\partial f_4}{\partial J} = 0 < \infty, \frac{\partial f_4}{\partial V} = 0 < \infty, \frac{\partial f_4}{\partial R} = 0 < \infty$$

$$\frac{\partial f_5}{\partial S} = 0 < \infty, \frac{\partial f_5}{\partial E} = 0 < \infty, \frac{\partial f_5}{\partial I} = \alpha < \infty, \frac{\partial f_5}{\partial Q} = \omega < \infty, \frac{\partial f_5}{\partial J} = -(r_1 - \mu - d_2) < \infty$$

$$\frac{\partial f_6}{\partial S} = 0 < \infty, \frac{\partial f_6}{\partial E} = 0 < \infty, \frac{\partial f_6}{\partial I} = 0 < \infty, \frac{\partial f_6}{\partial Q} = 0 < \infty, \frac{\partial f_6}{\partial J} = 0 < \infty, \frac{\partial f_6}{\partial V} = -[\theta + (\mu+\pi)\lambda - \mu] < \infty$$

$$\frac{\partial f_7}{\partial S} = 0 < \infty, \frac{\partial f_7}{\partial E} = 0 < \infty, \frac{\partial f_7}{\partial I} = r_2 < \infty, \frac{\partial f_7}{\partial Q} = 0 < \infty, \frac{\partial f_7}{\partial J} = r_1 < \infty, \frac{\partial f_7}{\partial V} = 0 < \infty, \frac{\partial f_7}{\partial R} = -\mu < \infty$$

Observed that all the partial derivatives of the model equation are continues and bounded in the interval, $0 < R < \infty$ by the lemma 1, there exists a unique solution of equation (2) in the region *R*.

By Lipchitz's condition, from the first equation in (2),

Let $f(t,x) = S_x, E_x, I_x, Q_x, J_x, V_x, R_x$ and $f(t,y) = S_y, E_y, I_y, Q_y, J_y, V_y, R_y$

$$\|f(t,x) - f(t,y)\| = \|[\theta V_x + (1-\omega Q_x) - (q+\mu+\lambda)S_x] - [\theta V_y + (1-\omega Q_y) - (q+\mu+\lambda)S_y]\|$$

$$\leq \|[V_x + Q_x - S_x]\|(\theta + 1 - \omega - (q + \mu + \lambda) - \|[V_y + Q_y - S_y]\|(\theta + 1 - \omega - (q + \mu + \lambda) -))$$

$$= \|(V_x + Q_x - S_x) - (V_y + Q_y - S_y)\|(\theta + 1 + \omega + (q + \mu + \lambda))$$

$$\leq K_1 \|(V_x + Q_x - S_x) - (V_y + Q_y - S_y)\| \quad (3)$$

For $K_1 = (\theta + 1 + \omega + (q + \mu + \lambda))$

Similarly from second equation in (2)

$$\| f(t,x) - f(t,y) \| = \|[(1-\pi)\lambda V_x + \lambda S_x - (p(1-\gamma) + \gamma + \mu)E_x] - [(1-\pi)\lambda V_y + \lambda S_y - (p(1-\gamma) + \gamma + \mu)E_y]\|$$

$$\leq \begin{matrix} \|[V_x + S_x - E_x]\|[(1+\pi)\lambda + \lambda + (p(1+\gamma) + \gamma + \mu)] - \\ \|V_y + S_y - E_y\|[(1+\pi)\lambda + \lambda + (p(1+\gamma) + \gamma + \mu)] \end{matrix}$$

$$\leq K_2 \|[V_x + S_x - E_x] - [V_y + S_y - E_y]\| \quad (4)$$

Where $K_2 = [(1+\pi)\lambda + \lambda + (p(1+\gamma) + \gamma + \mu)]$

Again from the third equation of (2)

$$\| f(t,x) - f(t,y) \| = \|[p(1-\gamma)E_x - (\alpha + \mu + d_1 I_x)] - [p(1-\gamma)E_y - (\alpha + \mu + d_1 I_y)]\|$$

$$\leq \|E_x - I_x\|[p(1-\gamma) + (\alpha + \mu + d_1)] - \|E_y - I_y\|[p(1-\gamma) + (\alpha + \mu + d_1)]$$

$$= \|[E_x - I_x] - [E_y - I_y]\|[p(1-\gamma) + (\alpha + \mu + d_1)]$$

$$\leq K_3 \|[E_x - I_x] - [E_y - I_y]\| \quad (5)$$

Where $K_3 = [p(1-\gamma) + (\alpha + \mu + d_1)]$

Similarly from the fourth equation of (2) we have

$$\| f(t,x) - f(t,y) \| = \|[\gamma E_x - (1 + \mu Q_x)] - [\gamma E_y - (1 + \mu Q_y)]\|$$

$$\leq \|[E_x - Q_x] - [E_y - Q_y]\|[\gamma + (1 + \mu)]$$

$$\leq K_4 \|[E_x - Q_x] - [E_y - Q_y]\| \quad (6)$$

For $K_4 = \gamma + (1 + \mu)$

Again considering the fifth equation of (2)

$$\| f(t,x) - f(t,y) \| = \|[\alpha I_x + \omega Q_x - (r_1 + \mu + d_2 J_x)] - [\alpha I_y + \omega Q_y - (r_1 + \mu + d_2 J_y)]\|$$

$$\leq \|(I_x + Q_x - J_x)\|[\alpha + r_1 + \mu + d_2] - \|(I_y + Q_y - J_y)\|[\alpha + r_1 + \mu + d_2]$$

$$= \left\| \left(I_x + Q_x - J_x\right) - \left(I_y + Q_y - J_y\right) \right\| \left[\alpha + r_1 + \mu + d_2\right]$$

$$\leq K_5 \left\| \left(I_x + Q_x - J_x\right) - \left(I_y + Q_y - J_y\right) \right\| \quad (7)$$

$K_5 = \left[\alpha + r_1 + \mu + d_2\right]$

From the sixth equation of (2) we have

$$\| f(t,x) - f(t,y) \| = \left\| \left[qS_x - \theta V_x - (1-\pi)\lambda V_x - \mu V_x\right] - \left[qS_y - \theta V_y - (1-\pi)\lambda V_y - \mu V_y\right] \right\|$$

$$\leq \left\| \left(S_x - V_x\right) \right\| \left(q + \left(\theta + (1-\pi)\lambda + \mu\right)\right) - \left\| \left(S_y - V_y\right) \right\| \left(q + \left(\theta + (1-\pi)\lambda + \mu\right)\right)$$

$$= \left\| \left(S_x - V_x\right) - \left(S_y - V_y\right) \right\| \left(q + \left(\theta + (1-\pi)\lambda + \mu\right)\right)$$

$$\leq K_6 \left\| \left(S_x - V_x\right) - \left(S_y - V_y\right) \right\| \quad (8)$$

Where $K_6 = \left(q + \left(\theta + (1-\pi)\lambda + \mu\right)\right)$

Lastly from the seventh equation in (2)

$$\| f(t,x) - f(t,y) \| = \left\| \left[r_1 J_x - \mu R_x\right] - \left[+ r_1 J_y - \mu R_y\right] \right\|$$

$$\leq \left\| \left[J_x - R_x\right] \right\| \left(r_1 + \mu\right) - \left\| \left[J_y - R_y\right] \right\| \left(+ r_1 + \mu\right)$$

$$= \left\| \left[J_x - R_x\right] - \left[J_y - R_y\right] \right\| \left(r_1 + \mu\right)$$

$$\leq K_7 \left[J_x - R_x\right] - \left[+ J_y - R_y\right] \quad (9)$$

For $K_7 = \left(r_1 + \mu\right)$

It implies that $\| f(x) - f(y) \| \leq K \| (x - y) \|$

Hence the model satisfies the Lipschitz condition with Lipschitz constants k_i

For $i = 1,2,3,4,5,6,7$

Therefore the existence and uniqueness of solution to the model equations has been established.

Positivity of solution

Lemma 2:

Let the initial data be $(S, E, I, Q, J, V, R)(0) \geq 0 \in \Phi$. Then, the solution set S, E, I, Q, J, V, R of equation (1), is positive for all $t > 0$.

Referring to lemma 2 and the system equation (1), the positivity of the solution was obtained

From the first equation of (1), we have

$$\frac{dS}{dt} = (1-\rho)\Lambda + \theta V + (1-\omega)Q - qs - \lambda s - \mu s \geq -s(q+\lambda+\mu) \quad (10)$$

Integrating equation (10), we have

$$\int \frac{ds}{s} \geq -\int (q+\lambda+\mu)dt$$

$$InS \geq -\int (q+\lambda+\mu)dt, S = e^{-(q+\lambda+\mu)t}$$

$$S(0) \geq S(0)e^{-(q+\lambda+\mu)t} \geq 0, \text{ since } (q+\lambda+\mu) > 0$$

From the second equation of (1), we have

$$\frac{dE}{dt} = (1-\pi)\lambda v + \lambda s - p(1-\gamma)E - \gamma E - \mu E \geq -[p(1-\gamma) - \gamma - \mu]E \quad (11)$$

$$E(t) \geq E(0)e^{-[p(1-\gamma)-\gamma-\mu]t} \text{ since } p(1-\gamma) - \gamma - \mu > 0$$

Similarly, From the third equation of (1), we have

$$\frac{dI}{dt} = p(1-\gamma)E + \phi_2 - \alpha I - (\mu + d_1)I$$

$$\geq -(\alpha + \mu + d_1)I \quad (12)$$

$$I(t) \geq I(0)e^{-(\alpha+\mu+d_1)t} \text{ since } \alpha + \mu + d_1 > 0$$

From the fourth equation of (1), we have

$$Q(t) \geq Q(0)e^{-(1-2\omega-\mu)t} \text{ since } (1-2\omega-\mu) > 0$$

From the fifth equation of (1), we have

$$J(t) \geq J(0)e^{-(r_1+\mu+d_2)t} \text{ since } (r_1+\mu+d_2) > 0$$

From the sixth equation of (1), we have

$$\frac{dV}{dt} = p\Lambda + qs - \theta v - (1-\pi)\lambda V - \mu V$$

$$\geq -[\theta - (1-\pi)\lambda - \mu]V \quad (13)$$

Integrating equation (13), we have

$$V(t) \geq V(0)e^{-(\theta-\lambda-\pi\lambda-\lambda)t} \text{ since } (\theta - \lambda - \pi\lambda - \mu) > 0$$

From the last equation of (1), we have

$$R(t) \geq R(0)e^{-\mu t} \text{ since } \mu > 0$$

Since the initial condition (S,E,I,Q,J,V,R)(0)≥ 0 are positive for all t->0, it implies that S(t), E(t), I(t), Q(t), J(t), V(t) and R(t) is positive for all t≥0, hence the prove have been established

Disease-Free Equilibrium (DFE) point

The disease-free equilibrium is a steady state solution by which there is no disease or any intervention and a closed population. That is infective classes equal to zero.

$$E = I = Q = J = V = R = 0$$

Theorem 1

A disease-free equilibrium state of the model (1) exist at the point

$$x_0 = (S, E, I, Q, J, V, R) = \left(\frac{(1-p)\Lambda}{\mu}, 0, 0, 0, 0, \frac{\rho\Lambda}{\theta+(1-\pi)\lambda+\mu} + \frac{q(1-p)\Lambda)}{\mu(\theta+(1-\pi)\lambda+\mu)}, 0\right)$$

Proof

At equilibrium state the rate of change of variable is equal to zero

$$S'(t) = E'(t) = I'(t) = Q'(t) = J'(t) = V'(t) = R'(t) = 0$$

$$S'(t) = (1-p)\Lambda + \theta V + (1-\omega)Q - qs - \mu s - \lambda s$$

$$S = \frac{((1-p)\Lambda)}{\mu} \quad (14)$$

In the presence of vaccination, implies $V \neq 0$

$$V'(t) = p\Lambda + qs - \theta v - (1-\pi)\lambda v - \mu v = 0$$

$$V = \frac{p\Lambda + qs}{\theta + (1-\pi)\lambda + \mu}$$

We have $V = \frac{\rho\Lambda}{\theta+(1-\pi)\lambda+\mu} + \frac{q(1-\rho)\Lambda}{\mu(\theta+(1-\pi)\lambda+\mu)}$ (15)

Since at the disease free equilibrium point the disease do not exist and assumed closed population. It implies E=I=Q=J=V=R=0. It has been proved as in equation (14). Also it has been proved that in the presence of vaccination (V) is given by equation (15). Therefore

$$x_0 = (S, E, I, Q, J, V, R) = \left(\frac{(1-p)\Lambda}{\mu}, 0, 0, 0, 0, \frac{\rho\Lambda}{\theta+(1-\pi)\lambda+\mu} + \frac{q(1-p)\Lambda)}{\mu(\theta+(1-\pi)\lambda+\mu)}, 0\right)$$

Feasible region (Invariant region)

Lemma 3

Let $X_1 = \left\{(S, E, I, Q, J, V, R) \epsilon R_+^7 : N \leq \frac{(1-\rho)\Lambda}{\mu}\right\}$

Be the feasible solutions set of model equation (1)

Proof,

$$N(t) = S(t) + E(t) + I(t) + Q(t) + J(t) + V(t) + R(t)$$

Implies

$$N'(t) = S'(t) + E'(t) + I'(t) + Q'(t) + J'(t) + V'(t) + R'(t)$$

$$N'(t) = (1-p)\Lambda + \phi_1 - \mu S - \mu E + \phi_2 - (\mu + d_1)I - \mu Q - (\mu + d_2)J - \mu V - \mu R \quad (16)$$

In the absence of immigration, TB induced death and vaccination implies $\phi_1 = \phi_2 = 0$ and $d_1 = d_2 = 0$ and $V = 0$ implies

$$\frac{dN}{dt} = (1-p)\Lambda - \mu(S + E + I + Q + J + V + R) \quad = (1-p)\Lambda - \mu N \quad (17)$$

$$\frac{dN}{dt} = (1-\rho)\Lambda - \mu N$$

$$\frac{dN}{dt} + \mu N = (1-\rho)\Lambda \quad (18)$$

Using linear method, we take the integrating factor $e^{\int \mu dt} = e^{\mu t}$

Multiply both side of (18) by $e^{\mu t}$

$$\left(\frac{dN}{dt} + \mu N\right)e^{\mu t} = ((1-\rho)\Lambda)e^{\mu t} \quad (19)$$

Integrating (19), we have

$$Ne^{\mu t} = (1-\rho)\Lambda \int e^{\mu t} dt$$

Finally we have

$$N(t) = N(0)e^{-\mu t} + \frac{(1-\rho)\Lambda}{\mu}(1 - e^{-\mu t})$$

$$N(t) \le N(0)e^{-\mu t} + \frac{(1-\rho)\Lambda}{\mu}(1 - e^{-\mu t}) \text{ as } t \to \infty$$

$$N(t) \le \frac{(1-\rho)\Lambda}{\mu}$$

Thus X_1 is a positively invariant set under the system in (1). Hence no solution path leave through any boundary of X_1, therefore solutions remain non-negative for non-negative initial conditions. Therefore the model equation (1) is mathematically and epidemiologically well posed.

Equilibrium points of the model

The equilibrium points of the system of non-linear ordinary differential equation describing the dynamics were obtained by setting the right hand side of the equation (1) to zero.

That is $S'(t) = E'(t) = I'(t) = Q'(t) = J'(t) = V'(t) = R'(t) = 0 \quad (20)$

Let $x_2 = (S^*, E^*, I^*, Q^*, J^*, V^*, R^*)$ be the equilibrium points.

The equilibrium point of the system of non-linear ordinary differential equation describing the dynamics were obtained by setting the right hand side of the equation (1) to zero.

That is $S'(t)=E'(t)=I'(t)=Q'(t)=J'(t)=V'(t)=R'(t)=0$

Thus at equilibrium point, the system of the equation becomes

$$\left.\begin{aligned}
&(1-\rho)\Lambda+\theta v+(1-\omega)Q-qs-\mu s-\lambda s=0\\
&(1-\pi)\lambda v+\lambda s-p(1-\gamma)E-\gamma E-\mu E=0\\
&p(1-\gamma)E+\phi_2-\alpha I-(\mu+d_1)I=0\\
&\gamma E-(1-\omega)Q-\omega Q-\mu Q+\phi_1=0\\
&\alpha I+\omega Q-r_1 J-(\mu+d_2)J=0\\
&\rho\Lambda+qs-\theta v-(1-\pi)\lambda v-\mu v=0\\
&r_1 J-\mu R=0
\end{aligned}\right\} \quad (21)$$

Solving for the state variables, we have the following equilibrium point of the system.

From the first equation (21)

$$S^{\bullet}=\frac{(1-\rho)\Lambda+\theta v+(1-\omega)Q}{q+\mu+\lambda} \quad (22)$$

From the second equation (21)

$$E^{\bullet}=\frac{\lambda s+(1-\pi)\lambda v}{p(1-\gamma)+\gamma+\mu} \quad (23)$$

From the third equation, we have

$$I^{\bullet}=\frac{p(1-\gamma)E+\phi_2}{(\alpha+\mu+d_1)} \quad (24)$$

Using the fourth equation of (21)

$$Q^{\bullet}=\frac{\gamma E+\phi_1}{1+\mu} \quad (25)$$

From the fifth equation of (21) we have

$$J^{\bullet}=\frac{\alpha I+\omega Q}{r_1+\mu+d_2} \quad (26)$$

From the sixth equation of (21) we have

$$V^{\bullet}=\frac{\rho\Lambda}{(\theta+\lambda(1-\pi)+\mu)}+\frac{qs}{(\theta+\lambda(1-\pi)+\mu)} \quad (27)$$

From the seventh equation of (21) we have

$$R^{\bullet} = \frac{r_1 J}{\mu} \tag{28}$$

The equilibrium points of the model system are

$$X_2 = \left(S^{\bullet}, E^{\bullet}, I^{\bullet}, Q^{\bullet}, J^{\bullet}, V^{\bullet}, R^{\bullet}\right)$$

Reproduction number

Here we state a theorem proposed by Diekmann *et al*,. (1990). It is assumed that each function is continuously differentiable at least twice in each variable.

Let $F_i(x)$ be the rate of appearance of new infections in compartment $i, V_i^+(x)$ be the rate of transfer of individuals into compartment i by all other means, and $V_i^-(x)$ be the rate of transfer of individuals out of compartment $ix_i = F_i(x) = f_i(x) - v_i(x), i = 1, \dots, n$

Where $v = v_i^- - v_i^+$

And the functions satisfy assumption (A1) through (A5) describe below. Since each function represents directed transfer of individuals, they are all non-negative.

$$A_1 \qquad if\ x \geq 0, then\ F_i, v_i^+, v_i^- \geq 0\ for\ i = 1, \dots, n,$$

If a compartment is empty, then there can be no transfer of individuals out of the compartment by death, infection, nor any other means. Thus

$$A_2 \qquad if\ x_i = 0\ then\ v_i^- = 0. in\ particular\ if\ x \epsilon x, then\ v_i^- = 0\ for\ i = 1, \dots, m$$

$$f = \frac{\partial f_i(x_0)}{\partial x_j} = \begin{bmatrix} \frac{\partial f_1(x_0)}{\partial E} & \frac{\partial f_1(x_0)}{\partial I} & \frac{\partial f_1(x_0)}{\partial Q} & \frac{\partial f_1(x_0)}{\partial J} \\ \frac{\partial f_2(x_0)}{\partial E} & \frac{\partial f_2(x_0)}{\partial I} & \frac{\partial f_2(x_0)}{\partial Q} & \frac{\partial f_2(x_0)}{\partial J} \\ \frac{\partial f_3(x_0)}{\partial E} & \frac{\partial f_3(x_0)}{\partial I} & \frac{\partial f_3(x_0)}{\partial Q} & \frac{\partial f_3(x_0)}{\partial J} \\ \frac{\partial f_4(x_0)}{\partial E} & \frac{\partial f_4(x_0)}{\partial I} & \frac{\partial f_4(x_0)}{\partial Q} & \frac{\partial f_4(x_0)}{\partial J} \end{bmatrix}$$

The next condition arises from the simple fact that the incidence of infection for uninfected compartment is zero. $A_3 \qquad f_i = 0 \ \ iff\ i > m.$

To ensure that the disease free subspace in invariant. We assume that if the population is free of disease then the population will remain free of disease.

$$A_4 \qquad if\ x \epsilon x, then\ F_i(x) = 0\ and\ v_i^+(x) = 0\ for\ i = 1, \dots, m$$

A_5 *if* $F_i(x)$ is set to zero, then all eigenvalues of $DF(x_0)$ have negative real parts.

If x_0 is a disease-free equilibrium of the model, x_0 is locally asymptotically stable if $R_0 < 1$, $unstable\ if\ R_0 > 1, where\ R_0\ is\ the\ reproduction\ number.$

We linearized equation (1) at X_0 by substituting the disease-free steady state. The result obtained will give us the transmission and transition matrices (Deikmann *et al.*, 2010).

f_i is the transmission matrix while V_i is the transition matrix.

$$V = \frac{\partial v_i(x_0)}{\partial x_j} = \begin{bmatrix} \frac{\partial v_1(x_0)}{\partial E} & \frac{\partial f_1(x_0)}{\partial I} & \frac{\partial v_1(x_0)}{\partial Q} & \frac{\partial v_1(x_0)}{\partial J} \\ \frac{\partial v_2(x_0)}{\partial E} & \frac{\partial f_2(x_0)}{\partial I} & \frac{\partial v_2(x_0)}{\partial Q} & \frac{\partial v_2(x_0)}{\partial J} \\ \frac{\partial v_3(x_0)}{\partial E} & \frac{\partial f_3(x_0)}{\partial I} & \frac{\partial v_3(x_0)}{\partial Q} & \frac{\partial v_3(x_0)}{\partial J} \\ \frac{\partial v_4(x_0)}{\partial E} & \frac{\partial v_4(x_0)}{\partial I} & \frac{\partial v_4(x_0)}{\partial Q} & \frac{\partial v_4(x_0)}{\partial J} \end{bmatrix}$$

$\frac{\partial F(x_0)}{\partial (X)}$ and $\frac{\partial V(x_0)}{\partial (X)}$, x_0 is the disease free equilibrium point and X represent the infective class.

The reproductive number is given by the dominant eigenvalue of the matrix

$$\left(\frac{\partial F(x_0)}{\partial (X)}\right)\left(\frac{\partial V(x_0)}{\partial (X)}\right)^{-1}$$

$$f_i = \begin{bmatrix} \lambda s \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\beta(I+\eta J)s}{N} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_i = -\begin{bmatrix} -p(1-\gamma)E - \gamma E - \mu E \\ p(1-\gamma)E + (\alpha + r_2 + \mu + d_1)I \\ \gamma E + (1-\omega) + \omega + \mu)Q \\ \alpha I - \omega Q + (r_1 + \mu + d_2)J \\ (\theta + \mu)V \end{bmatrix}$$

$$F = \frac{\partial f_i(x_0)}{\partial x_j} = \begin{bmatrix} \frac{\partial f_1(x_0)}{\partial E} & \frac{\partial f_1(x_0)}{\partial I} & \frac{\partial f_1(x_0)}{\partial Q} & \frac{\partial f_1(x_0)}{\partial J} \\ \frac{\partial f_2(x_0)}{\partial E} & \frac{\partial f_2(x_0)}{\partial I} & \frac{\partial f_2(x_0)}{\partial Q} & \frac{\partial f_2(x_0)}{\partial J} \\ \frac{\partial f_3(x_0)}{\partial E} & \frac{\partial f_3(x_0)}{\partial I} & \frac{\partial f_3(x_0)}{\partial Q} & \frac{\partial f_3(x_0)}{\partial J} \\ \frac{\partial f_4(x_0)}{\partial E} & \frac{\partial f_4(x_0)}{\partial I} & \frac{\partial f_4(x_0)}{\partial Q} & \frac{\partial f_4(x_0)}{\partial J} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\beta S}{N} & 0 & \frac{\beta n S}{N} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V = \frac{\partial v_i(x_0)}{\partial x_j} = \begin{bmatrix} \frac{\partial v_1(x_0)}{\partial E} & \frac{\partial f_1(x_0)}{\partial I} & \frac{\partial v_1(x_0)}{\partial Q} & \frac{\partial v_1(x_0)}{\partial J} \\ \frac{\partial v_2(x_0)}{\partial E} & \frac{\partial f_2(x_0)}{\partial I} & \frac{\partial v_2(x_0)}{\partial Q} & \frac{\partial v_2(x_0)}{\partial J} \\ \frac{\partial v_3(x_0)}{\partial E} & \frac{\partial f_3(x_0)}{\partial I} & \frac{\partial v_3(x_0)}{\partial Q} & \frac{\partial v_3(x_0)}{\partial J} \\ \frac{\partial v_4(x_0)}{\partial E} & \frac{\partial v_4(x_0)}{\partial I} & \frac{\partial v_4(x_0)}{\partial Q} & \frac{\partial v_4(x_0)}{\partial J} \end{bmatrix} = \begin{bmatrix} p(1-\gamma)+\gamma+\mu & 0 & 0 & 0 \\ -p(1-\gamma) & (\alpha+\mu+d_1) & 0 & 0 \\ -\gamma & 0 & (1+\mu) & 0 \\ 0 & -\alpha & -\omega & (r_1+\mu+d_2) \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \frac{1}{p(1-\gamma)+\gamma+\mu} & 0 & 0 & 0 \\ \frac{p(1-\gamma)}{(p(1-\gamma)+\gamma+\mu)(\alpha+\mu+d_1)} & \frac{1}{\alpha+\mu+d_1} & 0 & 0 \\ \frac{\gamma}{(p(1-\gamma)+\gamma+\mu)(1+\mu)} & 0 & \frac{1}{(1+\mu)} & 0 \\ \frac{\alpha p(1-\gamma)+\alpha p(1-\gamma)\mu+\omega\gamma\alpha+\omega\gamma\mu+\omega\gamma d_1}{(\alpha+\mu+d_1)(p(1-\gamma)+\gamma+\mu)(1+\mu)(r_1+\mu+d_2)} & \frac{\alpha}{(\alpha+\mu+d_1)(r_1+\mu+d_2)} & \frac{\omega}{(1+\mu)(r_1+\mu+d_2)} & \frac{1}{(r_1+\mu+d_2)} \end{bmatrix}$$

$$FV^{-1} = \begin{bmatrix} 0 & \frac{\beta S}{N} & 0 & \frac{\beta \eta S}{N} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{p(1-\gamma)+\gamma+\mu} & 0 & 0 & 0 \\ \frac{p(1-\gamma)}{(p(1-\gamma)+\gamma+\mu)(\alpha+\mu+d_1)} & \frac{1}{\alpha+\mu+d_1} & 0 & 0 \\ \frac{\gamma}{(p(1-\gamma)+\gamma+\mu)(1+\mu)} & 0 & \frac{1}{(1+\mu)} & 0 \\ \frac{\alpha p(1-\gamma)+\alpha p(1-\gamma)\mu+\omega\gamma\alpha+\omega\gamma\mu+\omega\gamma d_1}{(\alpha+\mu+d_1)(p(1-\gamma)+\gamma+\mu)(1+\mu)(r_1+\mu+d_2)} & \frac{\alpha}{(\alpha+\mu+d_1)(r_1+\mu+d_2)} & \frac{\omega}{(1+\mu)(r_1+\mu+d_2)} & \frac{1}{(r_1+\mu+d_2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\beta S p(1-\gamma)}{N(p(1-\gamma)+\gamma+\mu)(\alpha+\mu+d_1)} + \frac{\beta\eta S + (\alpha p(1-\gamma)+\alpha p(1-\gamma)\mu+\omega\gamma\alpha+\omega\gamma\mu+\omega\gamma d_1)}{(\alpha+\mu+d_1)(p(1-\gamma)+\gamma+\mu)(1+\mu)(r_1+\mu+d_2)} & \frac{\beta S}{N(\alpha+\mu+d_1)} + \frac{\beta\eta S\alpha}{N(\alpha+\mu+d_1)(r_1+\mu+d_2)} & \frac{\beta\eta S\omega}{N(1+\mu)(r_1+\mu+d_2)} & \frac{\beta\eta S}{N(r_1+\mu+d_2)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$FV^{-1} - \lambda I =$$

$$= \begin{bmatrix} \frac{\beta S p(1-\gamma)}{N(p(1-\gamma)+\gamma+\mu)(\alpha+\mu+d_1)} + \frac{\beta\eta S + (\alpha p(1-\gamma)+\alpha p(1-\gamma)\mu+\omega\gamma\alpha+\omega\gamma\mu+\omega\gamma d_1)}{(\alpha+\mu+d_1)(p(1-\gamma)+\gamma+\mu)(1+\mu)(r_1+\mu+d_2)} & \frac{\beta S}{N(\alpha+\mu+d_1)} + \frac{\beta\eta S\alpha}{N(\alpha+\mu+d_1)(r_1+\mu+d_2)} & \frac{\beta\eta S\omega}{N(1+\mu)(r_1+\mu+d_2)} & \frac{\beta\eta S}{N(r_1+\mu+d_2)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$FV^{-1} - \lambda I =$$

$$\begin{bmatrix} \frac{\beta S p(1-\gamma)}{N(p(1-\gamma)+\gamma+\mu)(\alpha+\mu+d_1)} + \frac{\beta\eta S + (\alpha p(1-\gamma)+\alpha p(1-\gamma)\mu+\omega\gamma\alpha+\omega\gamma\mu+\omega\gamma d_1)}{(\alpha+\mu+d_1)(p(1-\gamma)+\gamma+\mu)(1+\mu)(r_1+\mu+d_2)} - \lambda & \frac{\beta S}{N(\alpha+\mu+d_1)} + \frac{\beta\eta S\alpha}{N(\alpha+\mu+d_1)(r_1+\mu+d_2)} & \frac{\beta\eta S\omega}{N(1+\mu)(r_1+\mu+d_2)} & \frac{\beta\eta S}{N(r_1+\mu+d_2)} \\ 0 & 0-\lambda & 0 & 0 \\ 0 & 0 & 0-\lambda & 0 \\ 0 & 0 & 0 & 0-\lambda \end{bmatrix}$$

$$\left|FV^{-1}-\lambda I\right|=-\left(\frac{\beta Sp(1-\gamma)}{N(p(1-\gamma)+\gamma+\mu)(\alpha+\mu+d_1)}+\frac{\beta\eta S+(\alpha p(1-\gamma)+\alpha p(1-\gamma)\mu+\omega\gamma\alpha+\omega\gamma\mu+\omega\gamma d_1)}{(\alpha+\mu+d_1)(p(1-\gamma)+\gamma+\mu)(1+\mu)(r_1+\mu+d_2)}-\lambda\right)\lambda^3$$

The characteristics polynomial is

$$-\left(\frac{\beta Sp(1-\gamma)}{N(p(1-\gamma)+\gamma+\mu)(\alpha+\mu+d_1)}+\frac{\beta\eta S+(\alpha p(1-\gamma)+\alpha p(1-\gamma)\mu+\omega\gamma\alpha+\omega\gamma\mu+\omega\gamma d_1)}{(\alpha+\mu+d_1)(p(1-\gamma)+\gamma+\mu)(1+\mu)(r_1+\mu+d_2)}-\lambda\right)\lambda^3$$

Solving for λ, we obtained

$$\left[\lambda=0,\lambda=0,\lambda=0,\lambda=\frac{\beta Sp(1-\gamma)}{N(p(1-\gamma)+\gamma+\mu)(\alpha+\mu+d_1)}+\frac{\beta\eta S+(\alpha p(1-\gamma)+\alpha p(1-\gamma)\mu+\omega\gamma\alpha+\omega\gamma\mu+\omega\gamma d_1)}{(\alpha+\mu+d_1)(p(1-\gamma)+\gamma+\mu)(1+\mu)(r_1+\mu+d_2)}\right]$$

The eigenvalues of our characteristic equation are

$$\lambda_1=0,\lambda_2=0,\lambda_3=0,\lambda_4=\frac{\beta Sp(1-\gamma)}{N(p(1-\gamma)+\gamma+\mu)(\alpha+\mu+d_1)}+\frac{\beta\eta S+(\alpha p(1-\gamma)+\alpha p(1-\gamma)\mu+\omega\gamma\alpha+\omega\gamma\mu+\omega\gamma d_1)}{(\alpha+\mu+d_1)(p(1-\gamma)+\gamma+\mu)(1+\mu)(r_1+\mu+d_2)}-\lambda$$

The most dominant eigenvalue is λ_4. Therefore our reproductive number (R_0) is

$$R_0=\frac{\beta(1-\rho)\Lambda p(1-\gamma)}{\mu N((p(1-\gamma)+\gamma+\mu)(\alpha+\mu+d_1))}+\frac{\beta\eta(1-\rho)\Lambda[\alpha p(1-\gamma)+\alpha p(1-\gamma)\mu]}{\mu N[(\alpha+\mu+d_1)(p(1-\gamma)+\gamma+\mu)(1+\mu)(r_1+\mu+d_2)]}+\frac{\omega\gamma(\alpha+\mu+d_1)\beta\eta(1-\rho)\Lambda}{\mu N[p(1-\gamma)(\alpha+\mu+d_1)+\gamma+\mu(r_1+\mu+d_2)(1+\mu)]}$$

The basic reproductive number R_O is often considered as the threshold quantity that determines when infection of disease (TB) can invade and persist in a population.

Local Stability of the DFE (x_0)

Theorem: 2

The disease free equilibrium point of the model system (1) is

i. Locally asymptotically stable if $R_0<1$
ii. Unstable if $R_0>1$

If and only if Jacobian $J(x_0)$ at the equilibrium point x_0 of a square matrix $A=[A_{ij}]Ef^{n,n}$ has negative trace and positive determinant (Nyerere *et al.*, 2014). Here we take the partial differentiation of (1) with respect to S,E,I,Q,J,V,R at the disease-free equilibrium which gives us:

$$|J(x_0)| = \begin{bmatrix} -a & 0 & 0 & (1-\omega) & 0 & 0 & 0 \\ 0 & -b & 0 & 0 & 0 & (1-\pi)\lambda & 0 \\ 0 & p(1-\gamma) & -c & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & -d & 0 & 0 & 0 \\ 0 & 0 & \alpha & \omega & -e & 0 & 0 \\ q & 0 & 0 & 0 & 0 & -f & 0 \\ 0 & 0 & r_2 & 0 & r_1 & 0 & -q \end{bmatrix}$$

where $a = (q + \mu + \lambda)$ $b = (p(1-\gamma) + \gamma + \mu)$ $c = (\alpha + r_2 + \mu + d_1)$ d=(1+ μ)

$e = (r_1 + \mu + d_2)$ $f = (\theta + (1-\pi)\lambda + \mu)$ $g = (\mu)$

Now we find the determinant of $J(x_0)$ i. e.

$$|J(x_0)| = \begin{bmatrix} -a & 0 & 0 & (1-\omega) & 0 & 0 & 0 \\ 0 & -b & 0 & 0 & 0 & (1-\pi)\lambda & 0 \\ 0 & p(1-\gamma) & -c & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & -d & 0 & 0 & 0 \\ 0 & 0 & \alpha & \omega & -e & 0 & 0 \\ q & 0 & 0 & 0 & 0 & -f & 0 \\ 0 & 0 & r_2 & 0 & r_1 & 0 & -q \end{bmatrix} = -abcdefq + q^2\gamma ce - q^2\gamma ce\pi\lambda > 0$$

when the determinant is simplified we have

$|J(x_0)| = ceg(g\gamma - g\gamma\pi + g\gamma\omega + g\gamma\omega\pi - fdba) > 0$

where $a = (q + \mu + \lambda)$ $b = (p(1-\gamma) + \gamma + \mu)$ $c = (\alpha + r_2 + \mu + d_1)$ d=(1+ μ)

$e = (r_1 + \mu + d_2)$ $f = (\theta + (1-\pi)\lambda + \mu)$ $g = (\mu)$

Substituting a, b, c, d, e, f, g, we have

$|J(x_0)| = ceg(g\gamma - g\gamma\pi + g\gamma\omega + g\gamma\omega\pi - fdb = (\alpha + r_2 + \mu + d_1)(r_1 + \mu + d_2)g(g\gamma - g\gamma\pi + g\gamma\omega +$

$g\gamma\omega\pi - (\theta + (1-\pi)\lambda + \mu) = (1 + \mu)(p(1-\gamma) + \gamma + \mu)\,(q + \mu + \lambda)) > 0$ (29),

Expanding equation (29) and simplify, we have

$$(1 + \mu)(pq + q\gamma + q\mu + p\mu + \mu\gamma + \mu^2 + p\lambda + \lambda\gamma + \lambda\mu) > (1 + \mu)(pq\gamma + p\mu\gamma + p\lambda\gamma)$$

When there is no vaccine at birth and to the susceptible individuals, that is $q = p = 0$.

Implies $(1 + \mu)(\mu\gamma + \mu^2 + p\lambda + \lambda\gamma + \lambda\mu) > 0$

And the trace is obtained by adding the diagonal elements of the matrix

$$T_r\ J(x_0) = -[(q + \mu + \lambda) + (p(1-\gamma) + \gamma + \mu) + (\alpha + r_2 + \mu + d_1) + (1 + \mu) + (r_1 + \mu + d_2) + (\theta + (1-\pi)\lambda + \mu)4 + (\mu)] < 0$$

Here it is clear that both the determinant and the trace satisfied the theorem 2.

since the Jacobian matrix $J(x_o)$ have negative trace and have positive determinant, implies the local stability of the Disease Free Equilibrium point is locally asymptotically stable (LAS).

Numerical Experiment

In this section, we give the illustration of the analytic results of the work by carrying out numerical

simulations using (MATLAB R 2018a) and the values of the variables and parameters in tables 3 and 4

Table 3: Description of variables with their initial values

Variable	Description	Value
S(t)	Susceptible Individual	15000
E(t)	Exposed Individual	10000
I(t)	Infected Individual	5000
Q(t)	Quarantine Individual	2000
J(t)	Sanitarium Class	1500
V(t)	Vaccinated Individual	8000
R(t)	Recovered Individual	1300

Table 4: Description of parameters with their values

Parameters	Descriptions	Values	Reference
Λ	The Proportion of New born	0.1891	Estimated
μ	Natural mortality rate	0.2041	Andreir, 2007
ρ	Proportion of vaccination at birth	0.8	Estimated
$1-p$	Proportion of those not vaccinated at birth	0.97	Estimated
ϕ_1	The recruitment rate of susceptible immigrant	0.3	Estimated
d_2	TB induced death rate for individuals in the sanitarium	0.3	Estimated
d_1	TB induced death rate for infected individuals	0.365	Adetunde, 2007
β	per capita transmission rate from susceptible to exposed class.	0.35	Agusto,2009
ϕ_2	The proportion of vaccinated immigrants	0.2	Estimated
ω	Proportion of quarantined individual that are infected go for treatment	0.22	Estimated
$1-\omega$	Proportion of quarantine individuals that are not infected	0.78	Estimated
r_2	The recovery rate of infected individuals on self-cure or medication	0.2	Agusto, 2009
q	The rate at which susceptible individuals are vaccinated	0.2	Estimated
ϕ_3	The number of infected immigrants	0.21	Estimated
α	The rate at which infected individuals move to sanitarium	0.8	Estimated
θ	The rate at which new born are vaccinated	0.7	Estimated
r_1	The recovery rate of individuals in the sanitarium	0.9	Estimated
π	The efficacy of vaccine	0.8	Nyerere *et al.*, 2014

λ	The force of infection	0.042	Estimated
p	The rate of movement from expose to infected class	0.03	Nyerere *et al.*, 2014
γ	proportion of exposed individuals quarantined	0.6	Estimated
$1-\gamma$	Proportion of exposed individuals that are not quarantined	0.4	Estimated
η	probability of acquiring TB infections	0.3	Estimated

Simulation Results

The simulation results for our numerical experiments using the data provided or defined in the table 3 and 4 are as follows.

The graphs of some compartments at varying points $(\mathbf{0}, \mathbf{0.25}, \mathbf{0.5})$

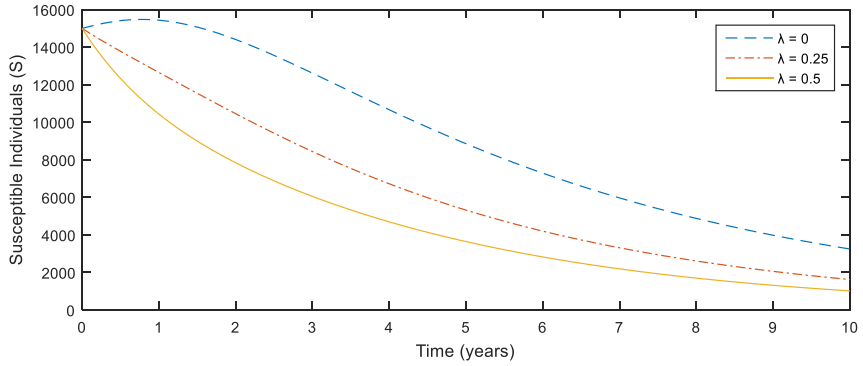

Figure 2: Susceptible Individuals using different levels of transmission rate $\lambda = 0, 0.25, 0.5$

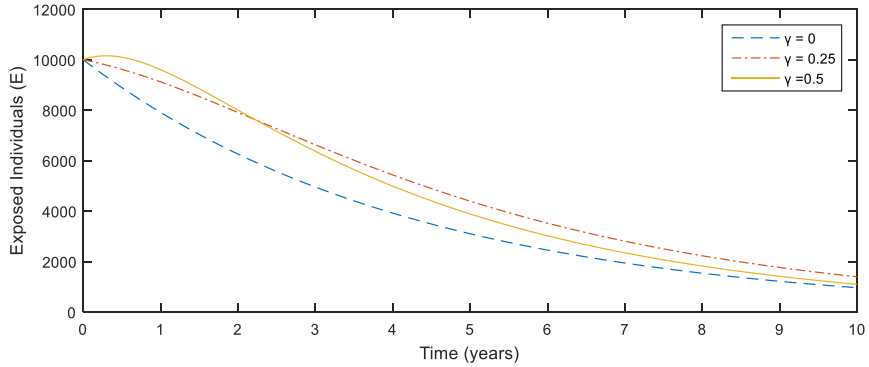

Figure 3: Exposed Individuals using different levels of proportion $\gamma = 0, 0.25, 0.5$

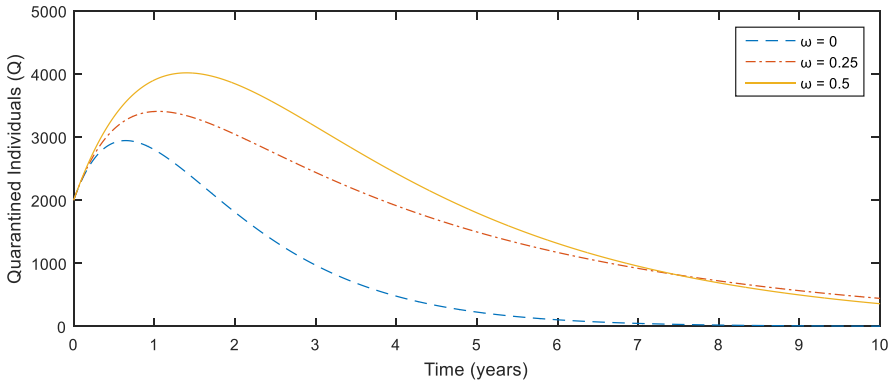

Figure 4: Quarantined Individuals using different levels of proportion $\omega = 0, 0.25, 0.5$

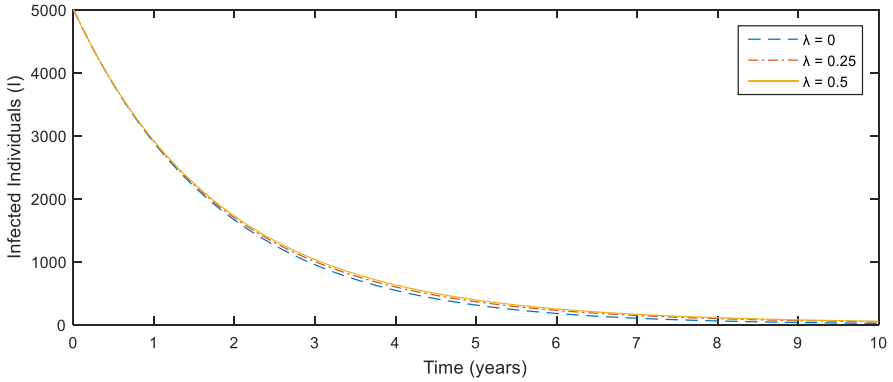

Figure 5: Infected Individuals using different levels of transmission rate $\lambda = 0, 0.25, 0.5$

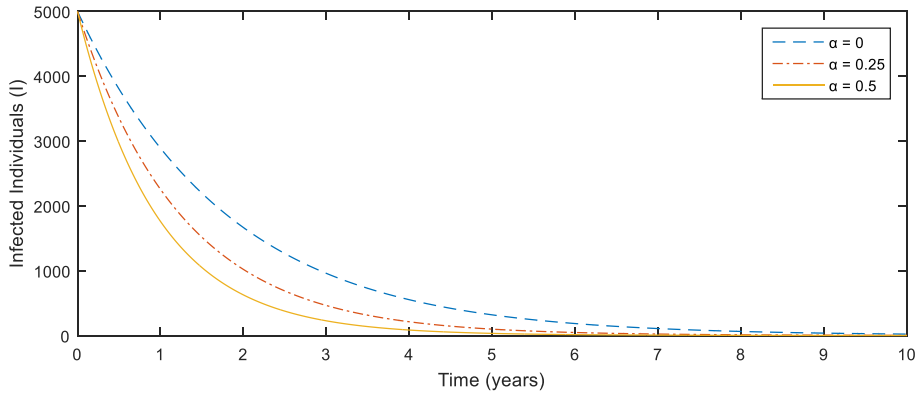

Figure 6: Infected Individuals using different levels of treatment rate $\alpha = 0, 0.25, 0.5$

Interpretation of the Graphical Results.
Figure 2 showed the dynamic of susceptible individuals using different levels of transmission rate over a period of ten years. When $\lambda = 0$, the susceptible individuals increases for the past two years, when $\lambda = 0.25$ the number of susceptible

individuals decreases gradually as well as when $\lambda = 0.5$ because some of them progress to latent stage after getting infected by infectious individuals.
From figure 3 we observed that the exposed individual's decreases gradually as the rate of quarantined increases. It may be due intensity of case detection and control of the immigrants at the border and treatment.

Also from figure 4, we observed that the quarantined individuals increases for first two years at different level of treatment (*w*) and decreases gradually as levels of treatment increases.
Figure 5 showed that the number of infected individuals decreases gradually with varying level of force of infection (λ) to zero at time t=10 years. This is because of the effectiveness of vaccine at birth and treatment in the sanitarium. Also due to the implementation of public campaign and case detection strategies.
Also, figure 6 showed that the number of infected individuals decreases to almost zero when the treatment rate $\alpha = 0.5$ after four years. This is because of the effectiveness of treatment given to the infected individuals

$$Z_{\rho}^{R_0} = \frac{\partial R_0}{\partial \rho} \times \frac{\rho}{R_0} = -1.22$$

$$Z_{\alpha}^{R_0} = \frac{\partial R_0}{\partial \alpha} \times \frac{\alpha}{R_0} = -0.14$$

$$Z_{\beta}^{R_0} = \frac{\partial R_0}{\partial \beta} \times \frac{\beta}{R_0} = -0.84$$

$$Z_{\omega}^{R_0} = \frac{\partial R_0}{\partial \omega} \times \frac{\omega}{R_0} = 0.16$$

$$Z_{d_2}^{R_0} = \frac{\partial R_0}{\partial d_2} \times \frac{d_2}{R_0} = -0.06$$

$$Z_{\Lambda}^{R_0} = \frac{\partial R_0}{\partial \Lambda} \times \frac{\Lambda}{R_0} = 0.31$$

$$Z_{p}^{R_0} = \frac{\partial R_0}{\partial p} \times \frac{p}{R_0} = 0.50$$

$$Z_{\mu}^{R_0} = \frac{\partial R_0}{\partial \mu} \times \frac{\mu}{R_0} = -0.64$$

Sensitivity analysis

To determine how best to reduce human mortality and morbidity due to TB, it is necessary to know the relative performance of different factors responsible for its transmission and prevalence. Knowing that disease transmission is directly related to reproduction number (R_0). We calculate the sensitivity indices of the reproduction number to the parameters in the model. This indices tells us how important is each parameter to disease transmission and prevalence. The essence of sensitivity analysis is to determine the robustness of model predictions to parameter and to discover which parameters have high impact on the reproduction number. The reproduction number is given by (R_0) and it depends on eight parameters(ρ , $d_2, \alpha, \beta, \omega, \Lambda, p, \mu$). We derive an analytical expression for its sensitivity to each parameter as follows (Chitnis, *et al*. 2008).

$$Z_{\rho}^{R_0} = \frac{\partial R_0}{\partial \rho} \times \frac{\rho}{R_0}$$

Applying this, using our R_0 and the values of the parameters as in table 5. We obtained the following sensitivity index.

Table 5: Parameter values for sensitivity analysis

parameter	Value	Reference
ρ	0.8	Estimated
d_2	0.3	Estimated
α	0.8	Estimated
β	0.35	Agusto, 2009
ω	0.22	Estimated
Λ	0.189	Estimated
p	0.03	Nyerere *et al.*, 2014
μ	0.204	Estimated

A positive index indicate that the value of R_0 increases as the parameter is increased while a negative index means the value of R_0 decreases as the parameters is increased.

Table 6: Sensitivity indices

parameter	Sensitivity index
ρ	-1.22
d_2	-0.06
α	-0.14
β	0.84
ω	0.16
Λ	0.30
p	0.50
μ	-0.64

Interpretation of the sensitivity indices

From table 5, the sensitivity indices shows that increasing (or decreasing) (β) the per capital transmission rate from susceptible to expose class, increases (or decreases) the reproduction number by 8.4%. Increasing (or decreasing) quarantine (ω) the quarantine individuals goes for treatment, increases (or decreases) the reproduction number by 1.6%. Increasing (or decreasing) birth rate (Λ), increases (or decreases) the reproduction number by 3%. Increasing (or decreasing) exposed individuals to infected class (p), increases (or decreases) the reproduction number by 5%.

The negative sign of the sensitivity index of R_0 with respect to ρ, d_2, α and μ means an inverse relationship between these parameters and R_0. The relationship implies increase (or decrease) in vaccination at birth (ρ) leads to approximately a 12.2% decrease (or increase) in R_0. Which means most people are vaccinated at birth to reduce the probability of being infected. Likewise increase (or decrease) in death in sanitarium (d_2) leads to approximately a 0.06% decrease (or increase) in R_0 which implies effectiveness of treatment and patients complete their treatment. Similarly increase (or decrease) in (α) rate of infected individuals goes for

treatment and (μ) natural death leads to approximately 1.4% and 6.4% decrease (or increase) in R_0 respectively. Therefore, to minimize TB transmission in the population, there is need for the combination of public campaign, case detection, vaccination, quarantine and sanitarium as control strategies to be implemented. This is due to the fact that, public campaign gives education and enlightenment on the transmission of TB, case detection and treatment reduces the progression rate to infectious stage, likewise vaccination and quarantine reduces the likelihood of an individual to get infected.

Conclusion

In this paper (work), a mathematical model for transmission of TB is formulated to determine the effects of the stated control strategies in the transmission of TB. The existence and uniqueness of solution to the differential equations was proved and the positivity of the variable was verified, in invariant region of the solution and the equilibrium points of the model were established, the reproductive number and the model equation is locally stable. Numerical experiments was carried out as well as the sensitivity analysis of the model parameters.

Recommendations

The recommendations are as follows:

i. Proper Education and sensitization on TB should be given to the general public by the government.
ii. The general public to acknowledge the role of case detection, vaccination and enlightenment on the dynamics of TB
iii. Health care workers should ensure that patients complete their treatment.

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