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Hybrid Block Method for the Solution of Nonlinear Systems of First-Order Ordinary Differential Equations

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Abstract
It is a known fact that systems of nonlinear ordinary differential equations have been known to be tedious to solve. In fact, some of the systems of nonlinear differential equations do not have closed form (exact) solutions. In view of the foregoing, this research is motivated by the need to derive a hybrid block method within a three-step integration interval $[x_n, x_{n+3}]$ for the solution of nonlinear system of equations. The formulation of the method was carried out via interpolation and collocation technique. The power series polynomial was adopted as basis function in deriving the method. Three off-grid points were carefully inserted within the three-step interval in order to guarantee a zero-stable method. The basic properties of the method were further analyzed. The results obtained showed that the method performed better than the ones results were compared with. The method is also efficient and computationally reliable.

Keywords: Block method, convergent, first-order, hybrid, nonlinear

Introduction

It is a known fact that nonlinear systems of differential equations find applications in various fields of human endeavor. This is because they are used to model different phenomenon in our today's world. Such equations appear not only in the physical sciences, but also in biology, engineering, management sciences, and all scientific disciplines that attempt to understand the world in which we live. It is also important to state that many of these equations that govern the physical world have no solution in closed form. Therefore, to find the answer to questions about the world in which we live, we must resort to solving these equations numerically.

In this paper, a hybrid block method is formulated for the solution of nonlinear first order systems of the form,

$$y'(x) = f(x, y),\ y(x_0) = y_0 \qquad (1)$$

where $f : \Re \times \Re^{2q} \to \Re^q; y, y_0 \in \Re^q$, and q is the dimension of the system. The function $f(x, y)$ is assumed to satisfy the Lipschitz condition stated in the Theorem 1.

Theorem 1 (Henrici, 1962)
Let $f(x, y)$ be a function, defined and continuous for all points (x, y) in the region D defined by $a \le x \le b, -\infty < y < \infty)$, a and b finite, and let there exist a constant L such that, for every x, y, y^* such that (x, y) and $f(x, y^*)$ are both in D,

$$\left| f(x, y) - f(x, y^*) \right| \le L \left| y - y^* \right| \qquad (2)$$

Then, if η is any given number, there exists a unique solution $y(x)$ of the initial value problem (1), where $y(x)$ is continuous and differentiable for all (x, y) in D. The requirement (2) is known as a Lipschitz condition and the constant L as a Lipschitz constant.

Over the years, quite a number of researchers have proposed methods for solving nonlinear systems of the form (1). These authors include Akinfenwa *et al.* (2020), Adesanya *et al.* (2018), Rufai *et al*. (2016), Adesanya *et al.* (2017), Ogunniran *et al.* (2020), Yakubu and Markus (2016), Akinnukawe and Muka (2020), Sunday *et al.* (2022), Kwari *et al.* (2023), among others. It is also important to give credit to researchers like Lambert (1973, 1991) and Fatunla (1980) who laid the foundation for deriving block methods for the solutions of differential equations of the form (1).

However, it is important to state that some of these methods have some setbacks ranging from large number of function evaluations, small convergence/implementation region to low order of accuracy. In view of these setbacks, this research is motivated by the need to address some of these setbacks by formulating a hybrid block method for the solution nonlinear systems of differential equations of the form (1).

The proposed method will address some of these setbacks by deriving a method that is capable of generating simultaneous numerical approximations at different grid points within the interval of integration. This advantage will enhance the accuracy of the method. Another advantage of the method is that it is less expensive in terms of the number of function evaluations compared to the conventional linear multistep and the Runge-Kutta methods. It also preserves the traditional advantage of one-step methods of being self-starting and permitting easy change of step-size during integration.

Some existing methods like predictor-corrector methods as well as hybrid block method also cater for some of the setbacks mentioned above. See the works of Yashkun and Aziz (2019), Adeyefa and Omole (2022), Soomro et al. (2022), among others.

The paper is structured as follows. In Section 2, the formulation of the method was presented. In the third section, the method was analyzed while in Section 4, numerical examples and discussion of results were presented. Conclusions were drawn in the fifth section.

Formulation of the Hybrid Block Method

Let the approximate solution to (1) be given by the following power series of degree 7

$$y(x)=\sum_{n=0}^{7}a_nx^n=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7 \quad (3)$$

Differentiating equation (3) gives the expression

$$y'(x)=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+6a_6x^5+7a_7x^6 \quad (4)$$

Substituting (4) into (1) gives,

$$f(x,y)=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+6a_6x^5+7a_7x^6 \quad (5)$$

Now, interpolating (3) at point x_{n+s}, $s=\frac{5}{2}$ and collocating (5) at points x_{n+r}, $r=0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3$, leads to a system of nonlinear equation of the form

$$XA=U \quad (6)$$

where

$$A=[a_0\ \ a_1\ \ a_2\ \ a_3\ \ a_4\ \ a_5\ \ a_6\ \ a_7]^T,\quad U=\left[y_n\ \ f_n\ \ f_{n+\frac{1}{2}}\ \ f_{n+1}\ \ f_{n+\frac{3}{2}}\ \ f_{n+2}\ \ f_{n+\frac{5}{2}}\ \ f_{n+3}\right]^T$$

$$X = \begin{bmatrix} 1 & x_{n+\frac{5}{2}} & x^2_{n+\frac{5}{2}} & x^3_{n+\frac{5}{2}} & x^4_{n+\frac{5}{2}} & x^5_{n+\frac{5}{2}} & x^6_{n+\frac{5}{2}} & x^7_{n+\frac{5}{2}} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} & 5x^4_{n+\frac{1}{2}} & 6x^5_{n+\frac{1}{2}} & 7x^6_{n+\frac{1}{2}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} & 7x^6_{n+1} \\ 0 & 1 & 2x_{n+\frac{3}{2}} & 3x^2_{n+\frac{3}{2}} & 4x^3_{n+\frac{3}{2}} & 5x^4_{n+\frac{3}{2}} & 6x^5_{n+\frac{3}{2}} & 7x^6_{n+\frac{3}{2}} \\ 0 & 1 & 2x_{n+2} & 3x^2_{n+2} & 4x^3_{n+2} & 5x^4_{n+2} & 6x^5_{n+2} & 7x^6_{n+2} \\ 0 & 1 & 2x_{n+\frac{5}{2}} & 3x^2_{n+\frac{5}{2}} & 4x^3_{n+\frac{5}{2}} & 5x^4_{n+\frac{5}{2}} & 6x^5_{n+\frac{5}{2}} & 7x^6_{n+\frac{5}{2}} \\ 0 & 1 & 2x_{n+3} & 3x^2_{n+3} & 4x^3_{n+3} & 5x^4_{n+3} & 6x^5_{n+3} & 7x^6_{n+3} \end{bmatrix}$$

Note that s and r are interpolation and collocation points respectively. Solving the system of nonlinear equation (6) by Gauss elimination method for the $a_j's,\ j = 0(1)7$ and substituting back into the power series (3) basis function gives a three-step hybrid block method as,

$$y(x) = \alpha_{\frac{5}{2}}(x) y_{n+\frac{5}{2}} + h\left(\sum_{j=0}^{3} \beta_j(x) f_{n+j} + \beta_{\frac{1}{2}}(x) f_{n+\frac{1}{2}} + \beta_{\frac{3}{2}}(x) f_{n+\frac{3}{2}} + \beta_{\frac{5}{2}}(x) f_{n+\frac{5}{2}} \right) \tag{7}$$

where

$$\left.\begin{aligned} &\alpha_{\frac{5}{2}}(x) = 1 \\ &\beta_0(x) = \frac{1}{120960}\begin{pmatrix} 1536t^7 - 18816t^6 + 94080t^5 - 246960t^4 \\ +363776t^3 - 296352t^2 + 120960t - 18575 \end{pmatrix} \\ &\beta_{\frac{1}{2}}(x) = -\frac{1}{5040}\begin{pmatrix} 384t^7 - 4480t^6 + 20832t^5 - 48720t^4 \\ +58464t^3 - 30240t^2 + 3625 \end{pmatrix} \\ &\beta_1(x) = \frac{1}{40320}\begin{pmatrix} 7680t^7 - 85120t^6 + 368256t^5 - 774480t^4 \\ +786240t^3 - 302400t^2 - 10625 \end{pmatrix} \\ &\beta_{\frac{3}{2}}(x) = -\frac{1}{945}\begin{pmatrix} 240t^7 - 2520t^6 + 10164t^5 - 19530t^4 \\ +17780t^3 - 6300t^2 + 625 \end{pmatrix} \\ &\beta_2(x) = \frac{1}{40320}\begin{pmatrix} 7680t^7 - 76160t^6 + 287616t^5 - 515760t^4 \\ +443520t^3 - 151200t^2 - 19375 \end{pmatrix} \\ &\beta_{\frac{5}{2}}(x) = -\frac{1}{5040}\begin{pmatrix} 384t^7 - 3584t^6 + 12768t^5 - 21840t^4 \\ +18144t^3 - 6048t^2 + 1175 \end{pmatrix} \\ &\beta_2(x) = \frac{1}{120960}\begin{pmatrix} 1536t^7 - 13440t^6 + 45696t^5 - 75600t^4 \\ +61376t^3 - 20160t^2 + 1375 \end{pmatrix} \end{aligned}\right\} \tag{8}$$

and t is given by

$$t = \frac{x - x_n}{h} \tag{9}$$

Evaluating (7) at $t = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ gives the discrete three-step hybrid block method,

$$\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}\begin{bmatrix} y_{n+\frac{1}{2}}\\ y_{n+1}\\ y_{n+\frac{3}{2}}\\ y_{n+2}\\ y_{n+\frac{5}{2}}\\ y_{n+3} \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{bmatrix}\begin{bmatrix} y_{n-\frac{1}{2}}\\ y_{n-1}\\ y_{n-\frac{3}{2}}\\ y_{n-2}\\ y_{n-\frac{5}{2}}\\ y_{n} \end{bmatrix} + h\begin{bmatrix} 0&0&0&0&0&\frac{19087}{120960}\\ 0&0&0&0&0&\frac{1139}{7560}\\ 0&0&0&0&0&\frac{137}{896}\\ 0&0&0&0&0&\frac{143}{945}\\ 0&0&0&0&0&\frac{3715}{24192}\\ 0&0&0&0&0&\frac{41}{280} \end{bmatrix}\begin{bmatrix} f_{n-\frac{1}{2}}\\ f_{n-1}\\ f_{n-\frac{3}{2}}\\ f_{n-2}\\ f_{n-\frac{5}{2}}\\ f_{n} \end{bmatrix}$$

$$+h\begin{bmatrix} \frac{2713}{5040} & -\frac{15487}{40320} & \frac{293}{945} & -\frac{6737}{40320} & \frac{263}{5040} & -\frac{863}{120960}\\ \frac{47}{63} & \frac{11}{2520} & \frac{166}{945} & -\frac{269}{2520} & \frac{11}{315} & -\frac{37}{7560}\\ \frac{81}{112} & \frac{1161}{4480} & \frac{17}{35} & -\frac{729}{4480} & \frac{27}{560} & -\frac{29}{4480}\\ \frac{232}{315} & \frac{64}{315} & \frac{752}{945} & \frac{29}{315} & \frac{8}{315} & -\frac{4}{945}\\ \frac{725}{1008} & \frac{2125}{8064} & \frac{125}{189} & \frac{3875}{8064} & \frac{235}{1008} & -\frac{275}{24192}\\ \frac{27}{35} & \frac{27}{280} & \frac{34}{35} & \frac{27}{280} & \frac{27}{35} & \frac{41}{280} \end{bmatrix}\begin{bmatrix} f_{n+\frac{1}{2}}\\ f_{n+1}\\ f_{n+\frac{3}{2}}\\ f_{n+2}\\ f_{n+\frac{5}{2}}\\ f_{n+3} \end{bmatrix} \tag{10}$$

Analysis of the Method

Some basic properties of the newly formulated method shall be analyzed in this section. These properties include order, error constant, consistency, zero-stability and region of absolute stability.

Definition 1: Order of a Block Method *(Fatunla, 1988)*

Let $y(x)$ be sufficiently differentiable, then the terms in a block method can be written as a Taylor series expansion about the point x as

Order and Error Constant of the Method

$$L\{y(x):h\} = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + \ldots + c_p h^p y^p(x) + c_{p+1} h^{p+1} y^{p+1}(x) + c_{p+2} h^{p+2} y^{p+2}(x) \tag{11}$$

where the constant coefficients $c_p,\ p = 0,1,2,\ldots$ are given by;

$$\left.\begin{array}{l} c_0 = \sum_{j=0}^{k} \alpha_j \\ c_1 = \sum_{j=0}^{k} \left(j\alpha_j - \beta_j \right) \\ \cdot \\ \cdot \\ \cdot \\ c_p = \sum_{j=0}^{k} \left[\frac{1}{q!} j^q \alpha_j - \frac{1}{(q-1)!} j^{q-1} \beta_j \right],\ q = 2,3,\ldots \end{array}\right\} \tag{12}$$

The method and its associated linear difference operators are said to have order p if $\overline{c_0} = \overline{c_1} = \overline{c_2} = \ldots = \overline{c_p} = 0 \text{ and } \overline{c}_{p+1} \neq 0$. The order is also defined as the largest positive real number p that quantifies the rate of convergence of a numerical approximation of a differential equation to that of the exact solution.

Definition 2: Error Constant (Fatunla, 1988)

The term $\overline{c}_{p+1}$ is called the error constant and implies that the local truncation error is given by,

$$t_{n+k} = \overline{c}_{p+1} h^{p+1} y^{(p+1)}(x_n) + O(h^{p+2}) \tag{13}$$

On the application of (12) on the newly formulated three-step hybrid block method (10), the expression below is obtained

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)^j}{j!} y_n^j - y_n - \frac{19087}{120960} hy_n' - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \begin{array}{l} \frac{2713}{5040}\left(\frac{1}{2}\right)^j - \frac{15487}{40320}(1)^j + \frac{293}{945}\left(\frac{3}{2}\right)^j \\ -\frac{6737}{40320}(2)^j + \frac{263}{5040}\left(\frac{5}{2}\right)^j - \frac{863}{120960}(3)^j \end{array} \right\} \\ \sum_{j=0}^{\infty} \frac{(1)^j}{j!} y_n^j - y_n - \frac{1139}{7560} hy_n' - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \begin{array}{l} \frac{47}{63}\left(\frac{1}{2}\right) + \frac{11}{2520}(1)^j + \frac{166}{945}\left(\frac{3}{2}\right)^j \\ -\frac{269}{2520}(2)^j + \frac{11}{315}\left(\frac{5}{2}\right)^j - \frac{37}{7560}(3)^j \end{array} \right\} \\ \sum_{j=0}^{\infty} \frac{\left(\frac{3}{2}\right)^j}{j!} y_n^j - y_n - \frac{137}{896} hy_n' - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \begin{array}{l} \frac{81}{112}\left(\frac{1}{2}\right)^j + \frac{1161}{4480}(1)^j + \frac{17}{35}\left(\frac{3}{2}\right)^j \\ -\frac{729}{4480}(2)^j + \frac{27}{560}\left(\frac{5}{2}\right)^j - \frac{29}{4480}(3)^j \end{array} \right\} \\ \sum_{j=0}^{\infty} \frac{(2)^j}{j!} y_n^j - y_n - \frac{143}{945} hy_n' - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \begin{array}{l} \frac{232}{315}\left(\frac{1}{2}\right)^j + \frac{64}{315}(1)^j + \frac{752}{945}\left(\frac{3}{2}\right)^j \\ +\frac{29}{315}(2)^j + \frac{8}{315}\left(\frac{5}{2}\right)^j - \frac{4}{945}(3)^j \end{array} \right\} \\ \sum_{j=0}^{\infty} \frac{\left(\frac{5}{2}\right)^j}{j!} y_n^j - y_n - \frac{3715}{24192} hy_n' - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \begin{array}{l} \frac{725}{1008}\left(\frac{1}{2}\right)^j + \frac{2125}{8064}(1)^j + \frac{125}{189}\left(\frac{3}{2}\right)^j \\ +\frac{3875}{8064}(2)^j + \frac{235}{1008}\left(\frac{5}{2}\right)^j - \frac{275}{24192}(3)^j \end{array} \right\} \\ \sum_{j=0}^{\infty} \frac{(3)^j}{j!} y_n^j - y_n - \frac{41}{280} hy_n' - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \begin{array}{l} \frac{27}{35}\left(\frac{1}{2}\right)^j + \frac{27}{280}(1)^j + \frac{34}{35}\left(\frac{3}{2}\right)^j \\ +\frac{27}{280}(2)^j + \frac{27}{35}\left(\frac{5}{2}\right)^j + \frac{41}{280}(3)^j \end{array} \right\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

Therefore,

$$\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = \bar{c}_5 = \bar{c}_6 = \bar{c}_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (15)$$

This implies that three-step hybrid block method is of the seventh order. That is,

$$p = \begin{bmatrix} 7 & 7 & 7 & 7 & 7 & 7 \end{bmatrix}^T \quad (16)$$

The error constant is given by

$$\bar{c}_8 = \begin{bmatrix} 4.4404\times10^{-5} \\ 3.3069\times10^{-5} \\ 3.9237\times10^{-5} \\ 3.3069\times10^{-5} \\ 4.4404\times10^{-5} \\ -1.2556\times10^{-5} \end{bmatrix}^T \quad (17)$$

Consistency of the Method

Definition 3: Consistency (Lambert, 1973)

A continuous linear multistep method is said to be consistent if its order $p \geq 1$.

Therefore, the method (10) is consistent since it has order $p \geq 1$.

Zero-Stability of the Method

Definition 4: Zero-Stability (Fatunla, 1988**)**

A continuous block method is said to be zero-stable if the roots $z_s,\ s = 1, 2, \ldots, n$ of the first characteristic polynomial denoted by $\bar{\rho}(z)$ satisfies $|z_s| \leq 1$ and every root with $|z_s| = 1$ has multiplicity not exceeding the order of the differential equation as $h \to 0$. The main consequence of zero-stability is to control the propagation of the error as the integration proceeds.

Applying Definition 4 on the method (10), the first characteristic polynomial is given by,

$$\rho(z) = \left| z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} z & 0 & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & 0 & -1 \\ 0 & 0 & z & 0 & 0 & -1 \\ 0 & 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & 0 & z-1 \end{vmatrix} = z^5(z-1)$$

Thus, solving for z in

$$z^5(z-1)=0 \tag{18}$$

gives $z_1 = z_2 = z_3 = z_4 = z_5 = 0$ *and* $z_6 = 1$. Hence, the hybrid block method (10) is zero-stable.

Convergence of the Method

Theorem 2 *(Dahlquist, 1963)*

The necessary and sufficient conditions for a linear multistep method to be convergent are that it be consistent and zero-stable.

The hybrid block method (10) is therefore convergent since it is consistent and zero-stable. See Theorem 2.

1.1. Region of Absolute Stability of the Method

Applying the boundary locus method, we obtain the stability polynomial for the hybrid block method (10) as,

$$\begin{aligned}\bar{h}(w) = -h^6\left(\frac{1}{448}w^5 - \frac{1}{448}w^6\right) - h^5\left(\frac{7}{320}w^6 + \frac{7}{320}w^5\right) - h^4\left(\frac{29}{240}w^5 - \frac{29}{240}w^6\right) \\ -h^3\left(\frac{7}{16}w^6 + \frac{7}{16}w^5\right) - h^2\left(\frac{25}{24}w^5 - \frac{25}{24}w^6\right) - h\left(\frac{3}{2}w^6 + \frac{3}{2}w^5\right) + w^6 - w^5\end{aligned} \tag{19}$$

The region of absolute stability of the hybrid block method is shown in Figure 1.

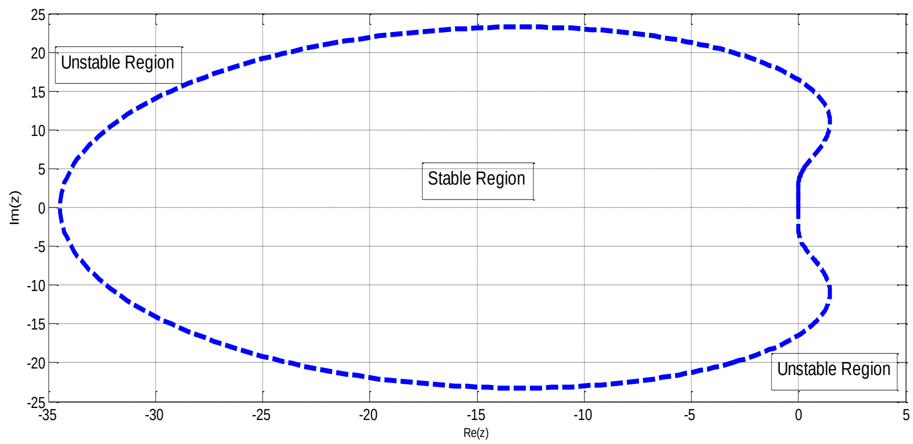

Figure 1: Stability region of the hybrid block method

The region of absolute stability of the hybrid block method is $A(\alpha)$-stable . Note that the stability region is the interior of the blue-coloured contour in Figure 1.

Results

Numerical Examples

The newly formulated hybrid block method will be applied in solving some nonlinear systems of equations of the form (1). This is aimed at testing the reliability of the new method. The following notations shall be used in the tables below:

h : step-size
HBM: Newly derived hybrid block method in equation (10)

Problem 1

Consider the nonlinear system,

$$\left.\begin{aligned} y_1' &= -2y_1 + y_2 + 2\sin x, \quad y_1(0) = 2 \\ y_2' &= 998y_1 - 999y_2 + 999(\cos x - \sin x), \quad y_2(0) = 3 \end{aligned}\right\} \tag{20}$$

whose exact solution is given by,

$$\left.\begin{aligned} y_1(x) &= 2e^{-x} + \sin x \\ y_2(x) &= 2e^{-x} + \cos x \end{aligned}\right\} \tag{21}$$

This system was solved by Akinnukawe *et al.* (2020) at the end point $x = 10$.

Problem 2

Consider the well-known nonlinear two-dimensional Kaps system

$$\left.\begin{aligned} y_1' &= -1002y_1 + 1000y_2^2, y_1(0) = 1 \\ y_2' &= y_1(x) - y_2(x)(1 + y_2(x)), y_2(0) = 1 \end{aligned}\right\} \tag{22}$$

defined in the range $0 \le x \le 20$, whose exact solution is given by

$$\left.\begin{aligned} y_1(x) &= e^{-2x} \\ y_2(x) &= e^{-x} \end{aligned}\right\} \tag{23}$$

Akinfenwa *et al.* (2020) solved this problem.

Problem 3

Consider the nonlinear Van der Pol system

$$\left.\begin{aligned} y_1' &= y_2, y_1(0) = 2 \\ y_2' &= -y_1 + \mu\left(1 - y_1^2\right)y_2, \ y_2(0) = 0 \end{aligned}\right\} \tag{24}$$

defined for $x \in [0, 70]$. This equation which is stiff in nature does not have a closed form solution. The Van der Pol equation replicates many phenomena in neurology, physics, biology, electronics and so on, Sunday et al. (2022). The system also serves as model in seismology, Cartwright (1999). The solution to (24) shall be computed at selected values of $\mu = 5$.

Problem 4

Consider the nonlinear chaotic Lorenz system

$$\left.\begin{aligned} y_1' &= \sigma(y_2 - y_1),\, y_1(0) = 1 \\ y_2' &= -y_1y_3 + \rho y_1 - y_2,\ y_2(0) = 5 \\ y_3' &= y_1y_2 - \tau y_3,\ y_3(0) = 10 \end{aligned}\right\} \qquad (25)$$

where the parameters σ, ρ and τ (all greater than zero) are proportional to the Prandtl number, Rayleigh number and some physical proportions of the region under consideration respectively. Equation (25) will be defined over $x \in [0,30]$ with $\sigma = 10$, $\rho = 28$ and $\tau = 8/3$ as considered by Lorenz (1963).

Discussion

The results presented in Table 1 clearly showed that the newly derived method performed better than the L-stable hybrid block method derived by Akinnukawe *et al.* (2020) at different values of the step-size. The method was further applied in solving the popular two-dimensional Kaps system in equation (22). From Table 2, it is obvious that the newly formulated method performed better than the seventh-order hybrid block method developed by Akinfenwa *et al.* (2020).

To further buttress the efficiency and accuracy of the hybrid block method derived, it was applied on some nonlinear systems that do not have exact solutions. That is, the Van der Pol and Lorenz systems. In such cases, the results obtained were compared with that of the inbuilt MATLAB solver (ode15s). Table 3 juxtaposes the numerical solution of the newly derived method and that of ode15s. The results from the table clearly show that the approximate solution of the new method (HBM) is in agreement with that of the ode15s.

Table 1. Comparison of absolute errors in HBM with that of Akinnukawe *et al.* (2020) for Problem 1

h	y_i	Error in HBM	Error in Akinnukawe *et al.* (2020)
0.25	y_1	1.19840×10^{-18}	4.50751×10^{-14}
	y_2	1.87291×10^{-18}	4.84057×10^{-14}
0.5	y_1	6.90917×10^{-18}	9.85878×10^{-14}
	y_2	6.18920×10^{-18}	9.81437×10^{-14}
1.0	y_1	4.90162×10^{-16}	9.45910×10^{-14}
	y_2	4.87265×10^{-16}	9.54792×10^{-14}
2.0	y_1	6.00932×10^{-16}	1.68310×10^{-13}
	y_2	6.12635×10^{-16}	1.68365×10^{-13}
4.0	y_1	2.78291×10^{-15}	2.21378×10^{-13}
	y_2	2.67432×10^{-15}	2.23044×10^{-13}
6.0	y_1	4.78210×10^{-15}	1.01363×10^{-13}

	y_2	4.98201×10^{-15}	1.01474×10^{-13}
8.0	y_1	2.00216×10^{-14}	1.93401×10^{-13}
	y_2	2.12562×10^{-14}	1.94650×10^{-13}
10.0	y_1	5.72863×10^{-14}	6.10623×10^{-13}
	y_2	5.65715×10^{-14}	6.09068×10^{-13}

Table 2. Comparison of the absolute errors in HBM with that of Akinfenwa *et al.* (2020) for Problem 2

h	N	y_i	Error in HBM	Error in Akinfenwa *et al.* (2020)
2.5	4	y_1	5.3190×10^{-16}	2.1670×10^{-09}
		y_2	5.9021×10^{-12}	1.3507×10^{-05}
1.25	8	y_1	2.1902×10^{-16}	2.3329×10^{-09}
		y_2	2.0192×10^{-12}	2.8914×10^{-05}
0.83333	12	y_1	3.0192×10^{-17}	2.3078×10^{-09}
		y_2	4.1028×10^{-13}	2.9695×10^{-05}
0.625	16	y_1	4.2617×10^{-18}	2.2987×10^{-09}
		y_2	4.9201×10^{-14}	2.9986×10^{-05}
0.5	20	y_1	1.1728×10^{-18}	2.2948×10^{-09}
		y_2	1.2781×10^{-14}	3.0115×10^{-05}

Table 3. Comparison of approximate solutions of Problem 3 using between HBM and ode15s at $h=0.1$

x	y_i	HBM	ode15s
1	y_1	-1.8650950811	-1.8650950571
	y_2	0.7524845612	0.7524845299
5	y_1	1.8985234702	1.8985234421
	y_2	-0.7289532600	-0.7289532451
10	y_1	1.7865365388	1.7865365103
	y_2	-0.8156276799	-0.8156276438
20	y_1	1.5075643401	1.5075643177
	y_2	-1.1911230101	-1.1911230003

Figure 2 shows the solution curves of the Van der Pol system in equation (24). The newly derived method was used to solve the system. From the solution curve, it is obvious that the solution of the newly derived method converge to that of the Matlab inbuilt solver (ode15s). This clearly shows that the method is computationally reliable.

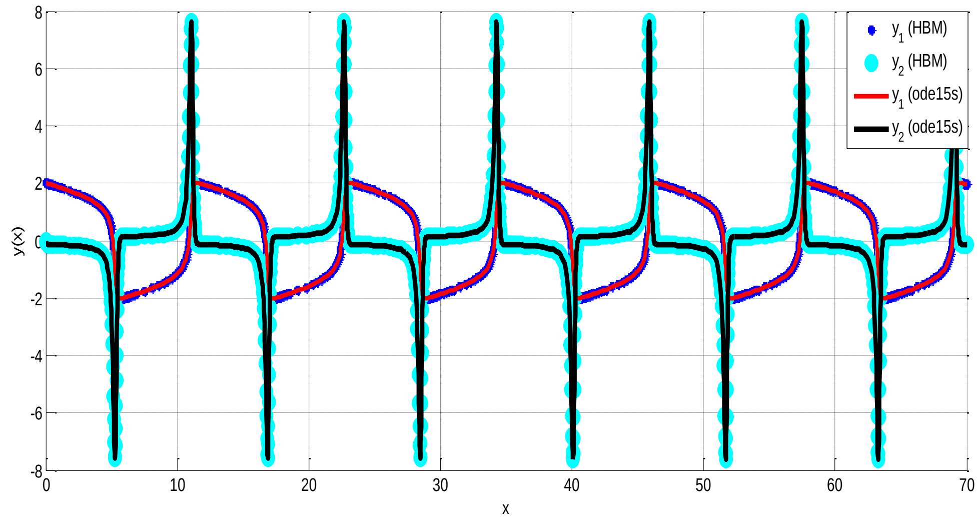

Figure 2: Solution curves for Problem 3 at $\mu = 5$

Figures 3-5 show the solution curves for the y_1, y_2 and y_3 components of the Lorenz system in equation (25). It is obvious from the three figures that the solution curves obtained using the newly derived method (HBM) converges to that of the ode15s. This implies that the method is computationally accurate and efficient.

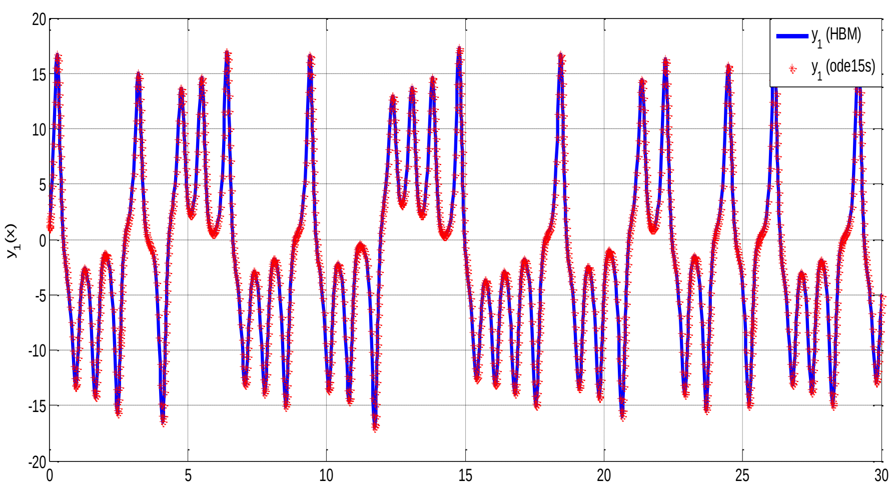

Figure 3: Solution curves of Problem 4 for y_1 component

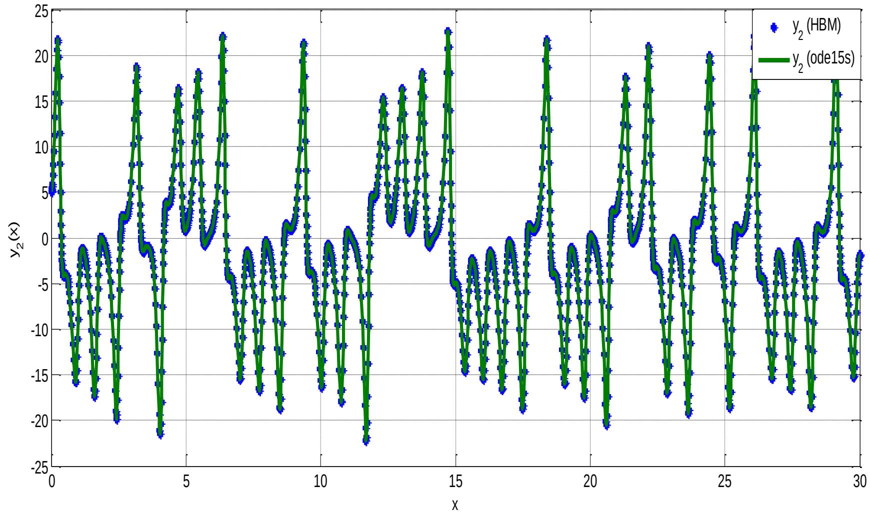

Figure 4: Solution curves of Problem 4 for y_2 component

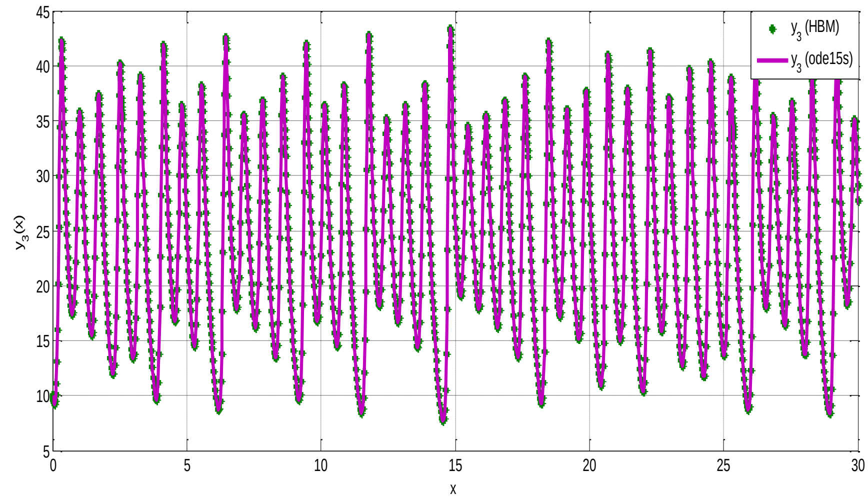

Figure 5.: Solution curves of Problem 4 for y_3 component

Conclusions

A hybrid block method has been formulated in this research for the solution of nonlinear systems of differential equations of the form (1). The paper further analyzes the basic properties of the method and found out that it is consistent, convergence and zero-stable. The results obtained on the application of the method of problems of the form (1) further showed that the method is computationally reliable.

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